KAZAKH NATIONAL UNIVERSITY named after AL-FARABI

Kenes B.Jakupov

CORRECTION OF CONTINUUM MECHANICS THEORETICAL PARADOXES (I)

Almaty
2009

UDC 532.516
ББК 22.5
D 40

Recommended for publishing
RSE Institute of mathematics, informatics and mechanics

R e v i e w e r s:
Doctor of Physical and Mathematical Sciences, Academician of NAC RK ***T.Sh.Kalmenov;***
Doctor of Physical and Mathematical Sciences, professor ***M.T.Zhienaliev***

Kenes B. Jakupov
D 40 Correction of continuum mechanics theoretical paradoxes – Almaty: publishing house Типография «К2», 2009 – 270p.
ISBN 978-601-0288-5

The monograph describes paradoxes of Stokes and Navier-Cauchy-Lame hypotheses. New equiations of viscous fluids dynamics, elasticity theory are theoretically justified. Current technologies of building effective numerical techniques for combined differential equation systems in true variables “velocity-pressure” are given in details.

The monograph is recommended for students, candidates for master's degree, post-graduate students, persons working for doctor's degree and researchers specialized in the sphere of fluid mechanics and elasticity theory.

Д $\frac{1603040000}{00(05)-09}$

ББК 22.5

ISBN 978-601-0288-5

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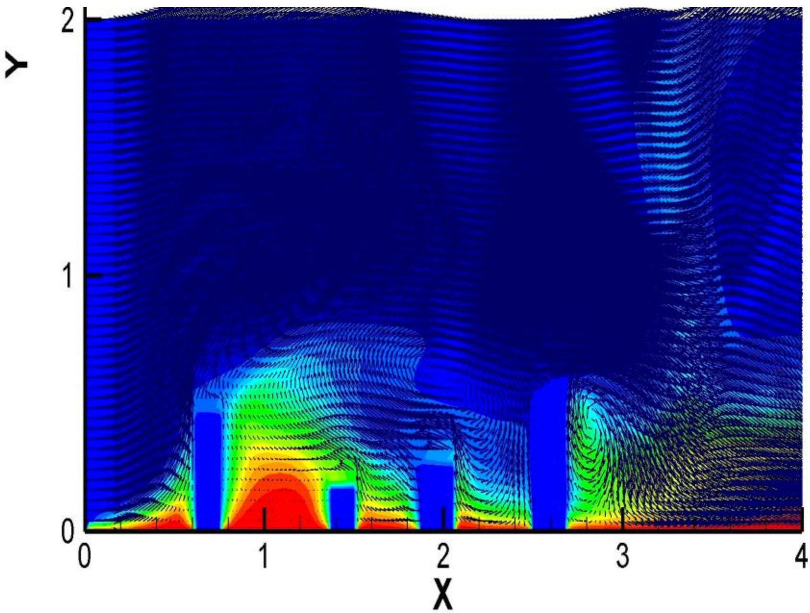

Residual concentrations of impurities (marked in red), demolished in the flow of air over the four buildings of varying heights. Wind flow is incident from the left side of buildings.

To the memory of professor Kh.I.Ibrashev
and Academician N.N.Yanenko
dedicated is this book

FOREWORD

The book was written on the basis of given by the author for many years lectures on “Continuum mechanics” and “Computational mechanics” at the faculty of mechanics and mathematics of the Kazakh National University named after *Al-Farabi*. Content of this book reflects up-to-date critical perception of a number of established fundamental principles and basic concepts of continuum mechanics, including hydrodynamics, and also individual aspects of practical applications. Estraordinary topicality of problems considered in this book is the only motive for combining discussion about these problems into one book in order to attract due attention to these problems, especially taking into account the fact that giving of many university courses, as a rule, starts from statement of these basic concepts and principles. Let us briefly indicate some of them. They are: inadequate application of *Euler-Leibniz* formula while deriving equations of dynamics in strains and other continuum equations; presentation of *Taylor* series (inexact differential) with dividing into symmetric and antisymmetric parts (which in some textbooks is formulated as the first *Helmholtz* theorem); incorrect interpretation of theorem on angular momentum change of particles system in relation to arbitrary volume of continuous medium, which resulted in fallacious theoretical statement about “symmetry” of strain tensor in continuous medium. *False simmetry of strain tensor* in particular, and included as a component into *Taylor’s* series, strain rate symmetric tensor were taken by *Stokes* as the basis for *hypothesis* about generalized *Newton* law, *fallacy* of which is being proved in this book. Therefore, formulated **in 1845** *Stokes* equations are not correct, due to this reason they may not be used as mathematical models of flows of compressible viscous gas, incompressible fluid with transient viscosity. Further on, derived in 1904 from these equations in relation to flows in stream boundary layer, *Prandtl* equations contradict to fundamental laws of physics: mass conservation law, second *Newton* law. Proposed, with hydrotechnic purposes simple formula of filtration by engineer *Darcy*, obtained incorrect interpretation distorting its primary physical essence, and in the result artificial “*Darcy* law” as a spacial filtering model was created. “*Darcy* law” also conflicts with *Newton* laws, energy conservation law, which excludes possibility of applying this pseudolaw as the adequate filtration model.

Existing from 1882 problem of closing averaged *Reynolds* equations for turbulent flows by engaging additional equations for second, third and higher moments, results, as is known, to divergent nonterminating chain of *Fridman-Keller* equations, other semiempirical models groundlessly use the whole number of constants and functional associations, in no way possessing universality, moreover, physical intensionality etc.

No doubt, from *the point of nonsymmetry of strain tensor,* necessity of revision of *Navier-Cauchy-Lame* equations in elasticity theory is logical.

Thanks to development of computational means, equations of continuum mechanics found wide range of applications in numerical experiments aimed at solving applied problems, and this was the basis for origination of its contemporary branch – *computational hydromechanics.* In connection with this, while stating fundamentals of the course of computational hydromechanics, it was first of all considered necessary to reasonably indicate *danger and inadmissibility* of applying *staggered grid* in numerical algorithms (for each equation its own grid is used, shifted for a half-step from the main grid). Insuperable defect of algorithms with staggered grid is the fact that number of unknowns exceeds the number of finite-difference equations, and defining of extra unknowns inevitably results in appering of unavoidable computational errors that distort desired solution of input equations. Along with this, for numerical solution of compressible gas equations, difference schemes are applied, and calculations under these schemes give *negative values of gas density*, which certainly indicates nonapplicability of such schemes.

Setting the above-mentioned problems and providing basis for possibility of their solving constitute content of this book.

It is a pleasant duty to express my gratitude to the participants of city seminars of mechanics, International scientific conferences and congresses on mathematics and mechanics, held at the Kazakh National University named by Al-Farabi in 2006-2009 years, to participants of All-Russian Conference on Mathematics and Mechanics, dedicated to 60th anniversary of Mechanics and Mathematics at Faculty of Tomsk State University on 22-24th of September in 2008 year in Tomsk City, for a discussion of the presented results.

Chapter 1. FALLACY OF THE *NAVIER-STOKES* EQUIATIONS. NEW EQUIATIONS OF VISCOUS GAS DYNAMICS

Equations of viscous fluids motion were obtained by French scientists: in 1821 - *Navier*, in 1831 - *Poisson* and in 1843 *Saint-Venant.* In 1845 the great English physicist *Stokes assumed* existence of linear dependence of strains from components of strain rates symmetric tensor and on their basis derived viscous gas dynamics equations, later called *Stokes* equations as opposed to *Navier* equation for incompressible viscous fluid with constant viscosity. Due to absence of proper theoretical justification of the given by *Stokes* relationship, *L.D.Landau* called it *Stokes hypothesis. Stokes* hypothesis leads to paradoxical phenomena that contradicts physics of strains, which caused doubts in correctness of deriving of *Stokes* equations on its basis. Thorough surveys revealed a vital error in the hypothesis. It is in the fact that *Stokes* from the *Taylor* series $\vec{v}(\vec{r}+\delta\vec{r},t)=\vec{v}(\vec{r},t)+\overline{S}\,\delta\vec{r}$, transformed to look like $\vec{v}(\vec{r}+\delta\vec{r},t)=\vec{v}(\vec{r},t)\ +\dot{S}\delta\vec{r}+\overset{\text{€}}{S}\delta\vec{r}$, given to it by *Helmholtz*, used for determination of viscous strains tensor only one part of displacement tensor $\overline{S}=\dot{S}+\overset{\text{€}}{S}$, in particular symmetric strain rates tensor $\dot{S}$, at this neglecting its antisymmetric part $\overset{\text{€}}{S}$, which characterizes motion component, in particular no less important rotary motion medium. It goes without saying, that such neglect is paradoxical and alogical. Error of *Stokes* hypothesis is to a certain degree connected with rooted in theoretical physics *erroneous postulate*, according to which *continuum strains tensor is always symmetrical*, (derivation of strain tensor symmetry from moment of momentum change theorem is contained in almost all textbooks and monographs /1/,/2/,/3/,/4/ etc.)

Insistent need in theoretical justification in the general case of *asymmetry of continuum strain tensor, including asymmetry of viscous strain tensor,* appeared*, and, as a result of all this, justification of Stokes equations fallacy*. Results of researches carried out in this Chapter are new equations of flow dynamics with variable viscosity with *Newton* nonsymmetric strain tensor.

It was easy to see that problems of *Stokes* hypothesis and symmetry of strain tensor are closely connected with paradoxical applications of fundamental formulae, in which connection their revision here is extremely needed and advisable.

§1. Paradoxes of *Euler, Leibniz* formulae

/1/ describes *hydrodynamic* derivation of individual time derivative for moving volume τ of continuum, limited by surface σ:

$$\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau = \frac{\partial}{\partial t}\iiint_{\tau}\Phi\delta\tau + \oint\!\!\int_{\sigma}\Phi(\vec{v},\vec{n})\delta\sigma \;, \qquad (1)$$

referred to in /1/ as *Euler formula*.

Hereinafter, according to /1/, the following denotations are used: symbol $\delta\vec{r} = \delta x\vec{i} + \delta y\vec{j} + \delta z\vec{k}$ - arbitrary infinitesimal segments, performed in space at the given moment of time in the given point of continuum, symbol $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$ - infinitesimal displacements of fluid elements, derivating within infinitesimal time interval *dt*, $\vec{v} = u\vec{i} + v\vec{j} + w\vec{k}$ - velocity vector, also $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. In /2/ the following formula is derived

$$\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau = \iiint_{\tau}\frac{\partial\Phi}{\partial t}\delta\tau + \oint\!\!\int_{\sigma}\Phi(\vec{v},\vec{n})\delta\sigma \,, \qquad (2)$$

which is used in /3/ as *Leibniz formula*. The question arises: which of these formulae is correct? Answer to this question may be given only on the basis of defining a derivative as function increment ratio limit to argument increment. For differentiable integrand $\Phi\vec{v}$ by *x, y, z,* according to *Ostrogradsky – Gauss* (divergence) theorem, is brought to volume integral:

$$\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau = \iiint_{\tau}\left(\frac{\partial\Phi}{\partial t} + div(\Phi\vec{v})\right)\delta\tau \qquad (3)$$

Further on, formula of velocity of volume dilatation of infinitesimal continuum volume $\delta\tau$ from /1/ will be used very often:

$$\frac{d\delta\tau}{dt} = \delta\tau \cdot div\vec{v} \qquad (4)$$

Rigorous deduction of the given formula is given in /1/, new proof (4) is given here in §2. Paradoxes concerned therewith, that subsist two approaches in choice of field of integration τ.

First approach. It was applied by *Sedov L.I.* in /2/, where in view of (4) consider, that size τ of continuum is *locomotive field*, that is time function $\tau = \tau(t)$, as sum $\tau = \iiint_\tau \delta\tau$ of individual sizes $\delta\tau = \delta\tau(t)$, which depend on time. Let in instant $t+\Delta t$ locomotive size τ occupies status $\tau' = \tau(t+\Delta t)$. Hence, *Sedov L.I.* in /2/ makes up increase ratio

$$\frac{d}{dt}\iiint_\tau \Phi\delta\tau = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\{\iiint_{\tau'}\Phi(x,y,z,t+\Delta t)\delta\tau - \iiint_\tau \Phi(x,y,z,t)\delta\tau(t)\},$$

whence *Leibniz formulae* comes in limit (2). But in this expression, which was made by *Sedov*, only function Φ and size $\tau' = \tau(t+\Delta t)$ taken on instant $t+\Delta t$, wasn't taken into account dependence $\delta\tau = \delta\tau(t+\Delta t)$. If take into account this dependence from time, that according to Sedov increase ratio must be constitute in form

$$\frac{d}{dt}\iiint_\tau \Phi\delta\tau = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big\{\iiint_{\tau'}\Phi(x,y,z,t+\Delta t)\delta\tau(t+\Delta t) - \qquad (5)$$
$$- \iiint_\tau \Phi(x,y,z,t)\delta\tau(t)\Big\},$$

whence expression comes in limit

$$\frac{d}{dt}\iiint_\tau \Phi\delta\tau = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big\{\iiint_{\tau'}\Phi(x,y,z,t+\Delta t)\delta\tau(t+\Delta t) - \iiint_\tau \Phi(x,y,z,t)\delta\tau(t)\Big\}$$
$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big\{\iiint_{\tau'}\Phi(x,y,z,t+\Delta t)\delta\tau(t+\Delta t) - \iiint_\tau \Phi(x,y,z,t+\Delta t)\delta\tau(t+\Delta t) +$$
$$+ \iiint_\tau \Phi(x,y,z,t+\Delta t)\delta\tau(t+\Delta t) - \iiint_\tau \Phi(x,y,z,t)\delta\tau(t)\Big\}$$
$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big\{\iiint_{\tau'-\tau}\Phi(x,y,z,t+\Delta t)\delta\tau(t+\Delta t) + \iiint_\tau \Phi(x,y,z,t+\Delta t)\delta\tau(t+\Delta t) -$$

$$-\iiint_{\tau}\Phi(x,y,z,t)\delta\tau(t)\Big\} = \oiint_{\sigma}\Phi(\vec{v},\vec{n})\delta\sigma + \iiint_{\tau}\frac{\partial}{\partial t}(\Phi\delta\tau), \tag{6}$$

in view of, that in field $\tau'-\tau$ elementary size equals $\delta\tau(t+\Delta t)=(\Delta\vec{r},\vec{n})\delta\sigma$, therefore arise speed $\lim\limits_{\Delta t\to 0}\dfrac{\Delta\vec{r}}{\Delta t}=\vec{v}$ и $\tau'-\tau\to\sigma$, $\vec{n}-$ort of outer normal to σ.

Well, from (6) follow formula

$$\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau = \iiint_{\tau}\frac{\partial}{\partial t}(\Phi\delta\tau) + \oiint_{\sigma}\Phi(\vec{v},\vec{n})\delta\sigma, \tag{7}$$

that, apparently, differ from formulas *Euler* (1) and *Leibniz* (2). And, if begin then and consider, that in locomotive size $\tau'=\tau(t+\Delta t)$ on instant $t+\Delta t$ and coordinates of particles *x, y, z* will trade places on $x(t+\Delta t), y(t+\Delta t), z(t+\Delta t)$, that increase ratio must look like

$$\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big\{\iiint_{\tau'}\Phi(x(t+\Delta t),y(t+\Delta t),z(t+\Delta t),t+\Delta t)\delta\tau(t+\Delta t) -$$
$$-\iiint_{\tau}\Phi(x,y,z,t)\delta\tau(t)\Big\},$$

whence after transformation arrives in limit

$$\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big\{\iiint_{\tau'}\Phi(x(t+\Delta t),y(t+\Delta t),z(t+\Delta t),t+\Delta t)\delta\tau(t+\Delta t) -$$
$$-\iiint_{\tau}\Phi(x(t+\Delta t),y(t+\Delta t),z(t+\Delta t),t+\Delta t)\delta\tau(t+\Delta t) +$$
$$+\iiint_{\tau}\Phi(x(t+\Delta t),y(t+\Delta t),z(t+\Delta t),t+\Delta t)\delta\tau(t+\Delta t) -$$
$$-\iiint_{\tau}\Phi(x,y,z,t)\delta\tau(t)\Big\}$$
$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big\{\iiint_{\tau'-\tau}\Phi(x(t+\Delta t),y(t+\Delta t),z(t+\Delta t),t+\Delta t)\delta\tau(t+\Delta t) +$$
$$+\iiint_{\tau}[\Phi(x(t+\Delta t),y(t+\Delta t),z(t+\Delta t),t+\Delta t)\delta\tau(t+\Delta t) - \Phi(x,y,z,t)\delta\tau(t)]\Big\}$$

$$= \oiint_{\sigma} \Phi(\vec{v},\vec{n})\delta\sigma + \iiint_{\tau} \frac{d}{dt}(\Phi\delta\tau) \qquad (8)$$

Well, from (8) follow other formula of time derivative

$$\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau = \iiint_{\tau}\frac{d}{dt}(\Phi\delta\tau) + \oiint_{\sigma}\Phi(\vec{v},\vec{n})\delta\sigma, \qquad (9)$$

that differ from formulas of *Euler* (1), *Leibniz* (2) and (7).

Second approach. It was applied by *Loitsyansky L.G.* in /1/. Consider, that size τ is *stated field* of continuum, that is not time function $\tau' = \tau = const$. Paradox consists in next, that field of integration τ must be sum of *individual* sizes $\delta\tau = \delta\tau(t)$, which depend on time in view (4) $\tau = \iiint_{\tau}\delta\tau = const$.

Increase ratio in second subsist will so

$$\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\iiint_{\tau}\Phi(x(t+\Delta t),y(t+\Delta t),z(t+\Delta t),t+\Delta t)\delta\tau(t+\Delta t) - \right.$$

$$\left. - \iiint_{\tau}\Phi(x,y,z,t)\delta\tau(t)\right\}, \qquad (10)$$

where field of integration not change in time in view $\tau' = \tau = const$. From (10) after analogous transformation arrive in limit

$$\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\iiint_{\tau}\Phi(x(t+\Delta t),y(t+\Delta t),z(t+\Delta t),t+\Delta t)\delta\tau(t+\Delta t) - \right.$$

$$\left. - \iiint_{\tau}\Phi(x,y,z,t)\delta\tau(t)\right\} = \iiint_{\tau}\lim_{\Delta t\to 0}\frac{1}{\Delta t}[\Phi(x(t+\Delta t),y(t+\Delta t),$$

$$z(t+\Delta t),t+\Delta t)\delta\tau(t+\Delta t) - \Phi(x,y,z,t)\delta\tau(t)] = \iiint_{\tau}\frac{d}{dt}(\Phi\delta\tau)$$

In result of stated size $\tau' = \tau = const$ arrive formula

$$\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau = \iiint_{\tau}\frac{d}{dt}(\Phi\delta\tau), \qquad (11)$$

that is possible let differentiation under sign of integral.

Thus, in depend on subsist to calculation total derivative by time, arrive 5 different expressions: (1), (2), (7), (9), (11).

The fact in /1/, /2/, /3/, /4/, /5/ and the like for opening expression $\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau$ - like, apply formulas (3) or (11), that in this textbooks, constitute foundation of **deductive method** of derivation of differential equations of dynamics, energy balance, continuity equation and others.

Exactly by means of deductive method came in *well-known* differential ratio for moment

$$[\vec{r},\rho\frac{d\vec{v}}{dt}]-[\vec{r},\rho\vec{F}]-\sum_{j=1}^{3}\frac{\partial}{\partial x_j}[\vec{r},\vec{\pi}_j]=0,$$

from which derive *mistaken* status about symmetry of strain tensor of continuum.

In inductive method derivatives $\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau$ **- like aren't applied and prove disparity zero:**

$$[\vec{r},\rho\frac{d\vec{v}}{dt}]-[\vec{r},\rho\vec{F}]-\sum_{j=1}^{3}\frac{\partial}{\partial x_j}[\vec{r},\vec{\pi}_j]\neq 0,$$

thereby prove status about dissymmetry of strain tensor of continuum.

§2.Paradoxes of deformation motion of medium elementary volume

As is generally known, strain rates symmetric tensor $\dot{S}$ /1/,/2/,/3/,/4 appears in continuum mechanics. Goal of the paragraph is proving that nonsymmetric displacement tensor $\overline{S}$ possesses the same properties. In the theory of elementary volume deformative motion and with the purpose of establishing viscous continuum rheological dynamics laws, *Taylor* series is widely used

$$\vec{v}(\vec{r}+\delta\vec{r},t)=\vec{v}(\vec{r},t)+\frac{\partial\vec{v}}{\partial x}\delta x+\frac{\partial\vec{v}}{\partial y}\delta y+\frac{\partial\vec{v}}{\partial z}\delta z\ ,$$

at the fixed moment of time *t.* (Strictly speaking, all theoretical mechanics uses *Taylor* series as principle of *virtual displacement* of a point). *Taylor* series in matrix-vectiral form looks as follows

$$\vec{v}(\vec{r}+\delta\vec{r},t)=\vec{v}(\vec{r},t)+\overline{S}\delta\vec{r}\ , \qquad (1)$$

which has a matrix hereinafter referred to as the *displacement tensor*:

$$\overline{S}=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}, \delta\vec{r}=\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}\begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}$$

Having denoted elements of this matrix

$$\overline{S}_{xx}=\partial u/\partial x, \overline{S}_{yx}=\partial u/\partial y, \overline{S}_{zx}=\partial u/\partial z, \overline{S}_{yy}=\partial v/\partial y, \overline{S}_{xy}=\partial v/\partial x,$$

$$\overline{S}_{zz}=\partial w/\partial z, \overline{S}_{yz}=\partial w/\partial y, \overline{S}_{xz}=\partial w/\partial x, \overline{S}_{zy}=\partial v/\partial z,$$

it is possible to represent displacement tensor, by analogy with known in hydromechanics *strain rates tensor* $\dot{S}$, as follows:

$$\overline{S}=\begin{pmatrix} \overline{S}_{xx}\overline{S}_{yx}\overline{S}_{zx} \\ \overline{S}_{xy}\overline{S}_{yy}\overline{S}_{zy} \\ \overline{S}_{xz}\overline{S}_{yz}\overline{S}_{zz} \end{pmatrix}$$

Following /1/, let’s introduce three velocities $\dot{e}_x, \dot{e}_y, \dot{e}_z$ of specific elongation of fluid elementary vectors $\delta\vec{r}_1(\delta x,0,0)$, $\delta\vec{r}_2(0,\delta y,0), \delta\vec{r}_3(0,0,\delta z)$, located along axes of rectangular coordinate system with beginning in some point *M*:

$$\dot{e}_x=\frac{1}{\delta x}\frac{d}{dt}(\delta x), \dot{e}_y=\frac{1}{\delta y}\frac{d}{dt}(\delta y), \dot{e}_z=\frac{1}{\delta z}\frac{d}{dt}(\delta z) \qquad (2)$$

and three velocities of quadrantal angles shift $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ between axes, which were equal to $\pi/2$ before deformation. Cosines of these angles are derived from scalar products

$$\cos\gamma_{xy}=(\delta\vec{r}_1,\delta\vec{r}_2)/(\delta x\delta y), \cos\gamma_{yz}=(\delta\vec{r}_2,\delta\vec{r}_3)/(\delta z\delta y), \qquad (3)$$

$$\cos\gamma_{zx}=(\delta\vec{r}_3,\delta\vec{r}_1)/(\delta x\delta z)$$

Time derivatives of these angles are denoted as

$$\dot{\varepsilon}_{xy} = -\partial\gamma_{xy} / \partial t, \dot{\varepsilon}_{yz} = -\partial\gamma_{yz} / \partial t, \dot{\varepsilon}_{zx} = -\partial\gamma_{zx} / \partial t$$

The following relation is assumed the basis for defining of kinematic meaning of components of displacement tensor $\bar{S}$

$$\frac{d}{dt}(\delta\vec{r}) = \delta(\frac{d\vec{r}}{dt}) = \delta\vec{v} \ , \tag{4}$$

which results from equalities:

$$\frac{d}{dt}(\delta\vec{r}) = \frac{d}{dt}(\vec{r}_1 - \vec{r}) = \frac{d\vec{r}_1}{dt} - \frac{d\vec{r}}{dt} = \vec{v}(\vec{r}_1) - \vec{v}(\vec{r}\ ) = \delta\vec{v}$$

Meaning $\vec{r}_1 = \vec{r} + \delta\vec{r}$, formula (1) is represented as

$$\delta\vec{v} = \bar{S}\delta\vec{r} \tag{5}$$

Substituting (4) into (5), obtained is equality

$$\frac{d}{dt}(\delta\vec{r}) = \bar{S}\delta\vec{r} \tag{6}$$

Assuming in it $\delta\vec{r}$ logically equal to $\delta\vec{r}_1(\delta x,0,0)$, $\delta\vec{r}_2(0,\delta y,0)$, $\delta\vec{r}_3(0,0,\delta z)$ and projecting to three axes, we'll find

$$\frac{d}{dt}(\delta x) = \bar{S}_{xx}\delta x \ , \ \frac{d}{dt}(\delta y) = \bar{S}_{yy}\delta y \ , \ \frac{d}{dt}(\delta z) = \bar{S}_{zz}\delta z \, ,$$

which, in accordance with (2), gives a relation

$$\bar{S}_{xx} = \dot{e}_x, \bar{S}_{yy} = \dot{e}_y, \bar{S}_{zz} = \dot{e}_z, \ \dot{e}_x = \partial u / \partial x, \dot{e}_y = \partial v / \partial y, \dot{e}_z = \partial w / \partial z \tag{7}$$

which results in equalities of diagonal displacement tensor component $\bar{S}$ corresponding to velocities of elongations of line elements, located on reference axes and having a beginning in the given point of flow.

Let's calculate derivatives by t from both parts of equalities (3):

$$-\sin\gamma_{xy}\frac{d\gamma_{xy}}{dt} = \frac{1}{\delta x\delta y}\frac{d(\delta\vec{r}_1,\delta\vec{r}_2)}{dt} + (\delta\vec{r}_1,\delta\vec{r}_2)\frac{d}{dt}(\frac{1}{\delta x\delta y}),$$

$$-\sin\gamma_{yz}\frac{d\gamma_{yz}}{dt} = \frac{1}{\delta z\delta y}\frac{d(\delta\vec{r}_2,\delta\vec{r}_3)}{dt} + (\delta\vec{r}_2,\delta\vec{r}_3)\frac{d}{dt}(\frac{1}{\delta z\delta y}),$$

$$-\sin\gamma_{zx}\frac{d\gamma_{zx}}{dt} = \frac{1}{\delta x\delta z}\frac{d(\delta\vec{r}_3,\delta\vec{r}_1)}{dt} + (\delta\vec{r}_3,\delta\vec{r}_1)\frac{d}{dt}(\frac{1}{\delta x\delta z}) \tag{8}$$

These equalities in /1/ are applied at the moment of time $t = t_0$, corresponding to initial ***strainless*** state of elementary volume, when

$$\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = \frac{\pi}{2}, (\delta\vec{r}_1, \delta\vec{r}_2) = 0, (\delta\vec{r}_2, \delta\vec{r}_3) = 0, (\delta\vec{r}_3, \delta\vec{r}_1) = 0$$

Will have from (8), because sinuses of right angles are equal to /1/:

$$\dot{\varepsilon}_{xy} = -\partial\gamma_{xy} / \partial t = \frac{1}{\delta x \delta y} \frac{d}{dt} (\delta\vec{r}_1, \delta\vec{r}_2),$$

$$\dot{\varepsilon}_{yz} = -\partial\gamma_{yz} / \partial t = \frac{1}{\delta y \delta z} \frac{d}{dt} (\delta\vec{r}_2, \delta\vec{r}_3),$$

$$\dot{\varepsilon}_{zx} = -\partial\gamma_{zx} / \partial t = \frac{1}{\delta z \delta x} \frac{d}{dt} (\delta\vec{r}_3, \delta\vec{r}_1) \qquad (9)$$

Using (6) and rules of calculation of triple scalar - vector products and products of matrix by vector, we have

$$\frac{d}{dt}(\delta\vec{r}_1, \delta\vec{r}_2) = \frac{d}{dt}(\delta\vec{r}_1)\delta\vec{r}_2 + \delta\vec{r}_1 \frac{d}{dt}(\delta\vec{r}_2) = ((\bar{S}\delta\vec{r}_1), \delta\vec{r}_2) + (\delta\vec{r}_1, (\bar{S}\delta\vec{r}_2)) =$$

$$= (\bar{S}\delta\vec{r}_1)_y \delta y + \delta x(\bar{S}\delta\vec{r}_2)_x = \bar{S}_{xy}\delta x\delta y + \bar{S}_{yx}\delta y\delta x = (\bar{S}_{xy} + \bar{S}_{yx})\delta x\delta y,$$

$$\frac{d}{dt}(\delta\vec{r}_2, \delta\vec{r}_3) = (\bar{S}_{yz} + \bar{S}_{zy})\delta z\delta y, \frac{d}{dt}(\delta\vec{r}_3, \delta\vec{r}_1) = (\bar{S}_{xz} + \bar{S}_{zx})\delta z\delta x,$$

Whence, due to (9) derived equalities

$$\dot{\varepsilon}_{xy} = \bar{S}_{xy} + \bar{S}_{yx}, \dot{\varepsilon}_{yz} = \bar{S}_{zy} + \bar{S}_{yz}, \dot{\varepsilon}_{zx} = \bar{S}_{xz} + \bar{S}_{zx} \qquad (10)$$

In /1/, /2/,/3/ and others, conforming to formula of the first Helmholtz theorem, strain rates tensor is introduced

$$\dot{S} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix} \qquad (11)$$

Along with this antisymmetric tensor is introduced

$$\tilde{S} = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{pmatrix}$$

In the overall equality $\overline{S} = \dot{S} + \tilde{S}$ is derived.

Using other representation (first *Helmholtz* theorem)

$$\frac{d}{dt}(\delta\vec{r}) = \vec{\omega} x \delta\vec{r} + \dot{S}\delta\vec{r}\ , \quad \vec{\omega} = \frac{1}{2} rot\vec{v}$$

in /1/ relations similar to (7) and (10) are obtained:

$$\dot{S}_{xx} = \dot{e}_x, \dot{S}_{yy} = \dot{e}_y, \dot{S}_{zz} = \dot{e}_z, \dot{e}_x = \partial u/\partial x, \dot{e}_y = \partial v/\partial y, \dot{e}_z = \partial w/\partial z,$$

$$\frac{d}{dt}(\delta\vec{r}_1, \delta\vec{r}_2) = (\dot{S}_{xy} + \dot{S}_{yx})\delta x\delta y, \frac{d}{dt}(\delta\vec{r}_2, \delta\vec{r}_3) = (\dot{S}_{yz} + \dot{S}_{yz})\delta z\delta y,$$

$$\frac{d}{dt}(\delta\vec{r}_3, \delta\vec{r}_1) = (\dot{S}_{zx} + \dot{S}_{xz})\delta z\delta x, \qquad (12)$$

Due to symmetry of strain rates tensor the following equalities are derived

$$\dot{S}_{xy} = \dot{S}_{yx} = \frac{1}{2}\dot{\varepsilon}_{xy}, \quad \dot{S}_{yz} = \dot{S}_{zy} = \frac{1}{2}\dot{\varepsilon}_{yz}, \quad \dot{S}_{zx} = \dot{S}_{xz} = \frac{1}{2}\dot{\varepsilon}_{zx} \qquad (13)$$

Paradoxical is the fact that from both (13) as well as (10) similar equalities are derived

$$\dot{\varepsilon}_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \dot{\varepsilon}_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \dot{\varepsilon}_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

despite of striking difference of tensors $\dot{S}$ and $\overline{S}$ from each other!

Let’s consider velocity of relative volume expansion of medium at its given point

$$\dot{\theta} = \frac{1}{\delta\tau}\frac{d}{dt}(\delta\tau),$$

$\delta\tau$ - elementary “liquid” volume of medium calculated by triple scalar - vector product of elementary coordinate vectors

$\delta\vec{r}_1(\delta x,0,0), \delta\vec{r}_2(0,\delta y,0), \delta\vec{r}_3(0,0,\delta z): \quad \delta\tau = (\delta\vec{r}_1, [\delta\vec{r}_2, \delta\vec{r}_3])$.

Calculating material time derivative, we obtain

$$\dot{\theta} = \frac{1}{\delta\tau}\frac{d}{dt}(\delta\tau) = \frac{1}{\delta\tau}\frac{d}{dt}(\delta\vec{r}_1, [\delta\vec{r}_2, \delta\vec{r}_3]) =$$

$$= \frac{1}{\delta\tau}\left(\frac{d}{dt}(\delta\vec{r}_1), [\delta\vec{r}_2, \delta\vec{r}_3]\right) + \frac{1}{\delta\tau}\left(\frac{d}{dt}(\delta\vec{r}_2), [\delta\vec{r}_3, \delta\vec{r}_1]\right) +$$

$$+ \frac{1}{\delta\tau}\left(\frac{d}{dt}(\delta\vec{r}_3), [\delta\vec{r}_1, \delta\vec{r}_2]\right)$$

For elementary parallelepiped $\delta\tau = \delta x\,\delta y\,\delta z$ and by known property of unit vectors on coordinate axes $\vec{i}, \vec{j}, \vec{k}$, and also due to representations

$\delta\vec{r}_1 = \delta x\vec{i} + 0\vec{j} + 0\vec{k}$, $\delta\vec{r}_2 = 0\vec{i} + \delta y\vec{j} + 0\vec{k}$, $\delta\vec{r}_3 = 0\vec{i} + 0\vec{j} + \delta z\vec{k}$

derived

$$\frac{[\delta\vec{r}_2, \delta\vec{r}_3]}{\delta\tau} = \frac{1}{\delta x}\left[\frac{\delta\vec{r}_2}{\delta y}, \frac{\delta\vec{r}_3}{\delta z}\right] = \frac{1}{\delta x}[\vec{j}, \vec{k}] = \frac{\vec{i}}{\delta x},$$

$$\frac{[\delta\vec{r}_3, \delta\vec{r}_1]}{\delta\tau} = \frac{\vec{j}}{\delta y}, \frac{[\delta\vec{r}_1, \delta\vec{r}_2]}{\delta\tau} = \frac{\vec{k}}{\delta z}$$

Due to which the previous equality will transform into the following

$$\dot{\theta} = \frac{1}{\delta\tau}\frac{d}{dt}(\delta\tau) = \frac{1}{\delta x}\left(\vec{i}, \frac{d}{dt}(\delta\vec{r}_1)\right) + \frac{1}{\delta y}\left(\vec{j}, \frac{d}{dt}(\delta\vec{r}_2)\right) + \frac{1}{\delta z}\left(\vec{k}, \frac{d}{dt}(\delta\vec{r}_3)\right)$$

Using again equality (6), according to which

$$\frac{d}{dt}(\delta\vec{r}_1) = \bar{S}\delta\vec{r}_1 \ , \ \frac{d}{dt}(\delta\vec{r}_2) = \bar{S}\delta\vec{r}_2 \ , \ \frac{d}{dt}(\delta\vec{r}_3) = \bar{S}\delta\vec{r}_3$$

we'll define desired relation $\dot{\theta}$ in the form of

$$\dot{\theta} = \frac{1}{\delta\tau}\frac{d}{dt}(\delta\tau) = \frac{1}{\delta x}(\vec{i}, \bar{S}\delta\vec{r}_1) + \frac{1}{\delta y}(\vec{j}, \bar{S}\delta\vec{r}_2) + \frac{1}{\delta z}(\vec{k}, \bar{S}\delta\vec{r}_3),$$

where scalar products are equal to

$$\frac{1}{\delta x}(\vec{i}, \bar{S}\delta\vec{r}_1) = \bar{S}_{xx} = \frac{\partial u}{\partial x}, \frac{1}{\delta y}(\vec{j}, \bar{S}\delta\vec{r}_2) = \bar{S}_{yy} = \frac{\partial v}{\partial y},$$

$$\frac{1}{\delta z}(\vec{k},\bar{S}\delta\vec{r}_3)=\bar{S}_{zz}=\frac{\partial w}{\partial z}$$

Finally derived

$$\dot{\theta}=\frac{1}{\delta\tau}\frac{d}{dt}(\delta\tau)=\bar{S}_{xx}+\bar{S}_{yy}+\bar{S}_{zz}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=div\,\vec{v} \qquad (14)$$

i.e. velocity of volume dilatation of elementary volume of medium at its given point is equal to divergence of a velocity vector at this point or sum of elongation of per unit length velocities

$$\dot{\theta}=\dot{e}_x+\dot{e}_y+\dot{e}_z$$

Hence, with the application of displacement tensor $\bar{S}$ established was the formula used above (4) **§1**. It directly results from (14):

$$\frac{1}{\delta\tau}\frac{d(\delta\tau)}{dt}=div\vec{v} \qquad (15)$$

or in *Euler* variables

$$\frac{\partial(\delta\tau)}{\partial t}+u\frac{\partial(\delta\tau)}{\partial x}+v\frac{\partial(\delta\tau)}{\partial y}+w\frac{\partial(\delta\tau)}{\partial z}=\delta\tau div\vec{v}$$

and expresses rapidity of change of elementary medium volume in time with motion assigned by it.

§3. Paradoxes of integral derivation of equation of dynamics continuum.

In /1/, /2/, /3/, /4/ apply integration by size τ, thus equations of dynamics continuum derivate by means of **deductive method**. Exactly applying of deductive method reduced to mistaken derivation about symmetry of strain tensor. Dispatch item is law of exchange impulse (or number of movements) for system points with masses m_i and speeds $\vec{v}_i$:

$$\frac{d}{dt}\sum_{i=1}^{N}\vec{p}_i=\sum_{i=1}^{N}\vec{f}_i \quad , \qquad (1)$$

where $\vec{p}_i=(\vec{v}_i m_i)$ - impulse (number of movements) particle with mass m_i, locomotive under operation of resultant force $\vec{f}_i$, in

which entered all forces, operate on particle with number *i*, as outward as inward forces of interaction particles between themselves. By summation these inward forces by third law of *Newton* are cancelled by pairs (see /7/).

In deductive method law (1) is applied to arbitrary size τ with surface σ, outward normal to which indicated with $\vec{n}$. Law of exchange impulse is written down for arbitrary size τ in form:

$$\frac{d}{dt}\iiint_{\tau}\vec{v}\rho\delta\tau = \iiint_{\tau}\vec{F}\rho\delta\tau + \oiint_{\sigma}\vec{\pi}_n\delta\sigma \qquad (2)$$

In /1/ transformation of left part (2) realized by formula (11) **§1**:

$$\frac{d}{dt}\iiint_{\tau}\vec{v}\rho\delta\tau = \iiint_{\tau}\frac{d}{dt}(\vec{v}\rho\delta\tau),$$

and in /2/ applied *Leibniz* formula. Apparently, in integral derivation of equation of dynamics continuum arise problems with calculations $\frac{d}{dt}\iiint_{\tau}\vec{v}\rho\delta\tau$ or derivative from moments $\frac{d}{dt}\iiint_{\tau}[\vec{r},\vec{v}\rho]\delta\tau$ and the like, on which was indicated in like 5 formulas in **§1.** Successful applying of *Leibniz* formula or (11) **§1** concerned with next, that equations of dynamics continuum had to correspond to theorem about exchange impulse or 2 law of *Newton*.

First, that paid attention, in superficial integral in (2) is taken into account only forces, operate on particle, situated on surface σ of size τ, that is *not taken into account inward efforts* $\oiint_{\sigma_{\delta\tau}}\vec{\pi}_n\delta\sigma$, *operate on particles of surface* $\sigma_{\delta\tau}$ *of individual size* $\delta\tau$ *,keeping mass* $\sum_i m_i = \delta m = \rho\delta\tau$ *of aggregate particles.* Taking into account these forces integral expression must have starting type

$$\frac{d}{dt}\iiint_{\tau}\vec{v}\rho\delta\tau = \iiint_{\tau}\vec{F}\rho\delta\tau + \iiint_{\tau}\oiint_{\sigma_{\delta\tau}}\vec{\pi}_n\delta\sigma \qquad (2a)$$

Further, size $\delta\tau_A$ *operate* on neighbouring size $\delta\tau_B$ with force $\vec{f}_A = \oint\!\!\oint_{\sigma_{\delta\tau_A}} \vec{\pi}_n \delta\sigma$ and, vice versa, size $\delta\tau_B$ *operate with force* $\vec{f}_B = \oint\!\!\oint_{\sigma_{\delta\tau_B}} \vec{\pi}_n \delta\sigma$ on size $\delta\tau_A$, than by 3 law of *Newton* take place equality

$$\iiint_{\tau} \oint\!\!\oint_{\sigma_{\delta\tau}} \vec{\pi}_n \delta\sigma = \oint\!\!\oint_{\sigma} \vec{\pi}_n \delta\sigma, \qquad (*)$$

that is so have to be basis (2). From rectilineal set expression (2a), if apply Leibniz formula to left part or formula (11) **§1,** derive equation of dynamics continuum in efforts, that is not necessary transition to formula (2).

In theorem about exchange of number of movements (impulse), which applied in /1/ to size τ like manner (2):

$$\frac{d}{dt}\iiint_{\tau}[\vec{r},\vec{v}\rho\delta\tau] = \iiint_{\tau}[\vec{r},\vec{F}\rho\delta\tau] + \oint\!\!\oint_{\sigma}[\vec{r},\vec{\pi}_n\delta\sigma] \qquad (3)$$

equality like (*) isn't take place. In accordance with aforecited reasonings rectilineal set theorem about exchange of moment of number of movements take type

$$\frac{d}{dt}\iiint_{\tau}[\vec{r},\vec{v}\rho\delta\tau] = \iiint_{\tau}[\vec{r},\vec{F}\rho\delta\tau] + \iiint_{\tau}[\vec{r},\oint\!\!\oint_{\sigma_{\delta\tau}}\vec{\pi}_n\delta\sigma] \qquad (3a)$$

From (3) derives equality, used in /1/ and other,

$$[\vec{r},\rho\frac{d\vec{v}}{dt}] - [\vec{r},\rho\vec{F}] - \sum_{j=1}^{3}\frac{\partial}{\partial x_j}[\vec{r},\vec{\pi}_j] = 0$$

with one's following from here oppositions about *symmetry of strain tensor of effort* continuum. From rectilineal set expression (3a) prove ratio

$$[\vec{r},\rho\frac{d\vec{v}}{dt}] - [\vec{r},\rho\vec{F}] - [\vec{r},\sum_{j=1}^{3}\frac{\partial\vec{\pi}_j}{\partial x_j}] = 0,$$

from which follows equation of dynamics in efforts

$$\rho \frac{d\vec{v}}{dt} = \rho \vec{F} + \sum_{j=1}^{3} \frac{\partial \vec{\pi}_j}{\partial x_j},$$

therefore, don’t question about *symmetry* of strain tensor of efforts, that took place in wrong formula (3).

Other *paradox* consists in next, that for derivation of fundamental mechanics equations of continuum, not any necessary in integral formulas like (2), (3). Show this.

Let mass of individual size $\delta\tau$ equals $\delta m = \rho\delta\tau$, $\vec{F}$ - density of body forces, $\vec{\pi}_n$ - effort, $\rho -$ density of medium. Theorem about exchange of impulse necessary formulate directly for size $\delta\tau$, take into account, that main surface force, operating on surface $\sigma_{\delta\tau}$ of size $\delta\tau$ equals $\oint\!\!\int_{\sigma_{\delta\tau}} \vec{\pi}_n \delta\sigma$, main body force, operate on size $\delta\tau$, equals $\vec{F}\delta m$:

$$\frac{d}{dt}(\vec{v}\delta m) = \vec{F}\delta m + \oint\!\!\int_{\sigma_{\delta\tau}} \vec{\pi}_n \delta\sigma \qquad (4)$$

Further by theorem by *Ostrogradsky* – *Gauss* make transition to volumetrical integral and smallness of individual size $\delta\tau$ prove

$$\iiint_{\delta\tau} \left(\frac{\partial \vec{\pi}_x}{\partial x} + \frac{\partial \vec{\pi}_y}{\partial y} + \frac{\partial \vec{\pi}_z}{\partial z}\right)\delta\tau' = \left(\frac{\partial \vec{\pi}_x}{\partial x} + \frac{\partial \vec{\pi}_y}{\partial y} + \frac{\partial \vec{\pi}_z}{\partial z}\right)\iiint_{\delta\tau} \delta\tau' =$$

$$= \left(\frac{\partial \vec{\pi}_x}{\partial x} + \frac{\partial \vec{\pi}_y}{\partial y} + \frac{\partial \vec{\pi}_z}{\partial z}\right)\delta\tau,$$

therefore, equality takes place

$$\oint\!\!\int_{\sigma_{\delta\tau}} \vec{\pi}_n \delta\sigma = \left(\frac{\partial \vec{\pi}_x}{\partial x} + \frac{\partial \vec{\pi}_y}{\partial y} + \frac{\partial \vec{\pi}_z}{\partial z}\right)\delta\tau \qquad (5)$$

Substitute (5) and $\delta m = \rho\delta\tau$ in (4), find

$$\frac{d}{dt}(\vec{v}\rho\delta\tau) = \vec{F}\rho\delta\tau + (\frac{\partial\vec{\pi}_x}{\partial x} + \frac{\partial\vec{\pi}_y}{\partial y} + \frac{\partial\vec{\pi}_z}{\partial z})\delta\tau \qquad (6)$$

Left part (6) is transformed on base of formula (4) **§1** and equations of continuity $d\rho / dt + \rho div\vec{v} = 0$:

$$\rho\frac{d\vec{v}}{dt}\delta\tau + \rho\vec{v}\,\delta\tau div\vec{v} + \vec{v}\,\delta\tau\frac{d\rho}{dt} =$$

$$= \vec{F}\rho\delta\tau + (\frac{\partial\vec{\pi}_x}{\partial x} + \frac{\partial\vec{\pi}_y}{\partial y} + \frac{\partial\vec{\pi}_z}{\partial z})\,\delta\tau\ ,$$

where from after cancellation $\delta\tau$ prove classic equation of dynamic continuum in efforts

$$\rho\frac{d\vec{v}}{dt} = \rho\vec{F} + \frac{\partial\vec{\pi}_x}{\partial x} + \frac{\partial\vec{\pi}_y}{\partial y} + \frac{\partial\vec{\pi}_z}{\partial z}$$

§4. Inductive approach

Inductive approach (from the particular to the general) is free from the given above disadvantages because notion of continuum *particle* is used. "Liquid" individual volume $\delta\tau$ is formed by these particles with masses m_i and velocities $\vec{v}_i$. (If inductive approach was used from the very beginning of development of continuum mechanics, there would be no problem of strain tensor symmetry! In this connection this physical approach is detailed here.)

It is assumed that due to medium continuity, $\delta\tau$ containes the sum of particles $\sum_i m_i = \delta m$, weight-average velocity $\vec{v}$ of volume $\delta\tau$ is defined as relations /2/

$\vec{v} = \sum_i m_i\vec{v}_i / \sum_i m_i = \sum_i m_i\vec{v}_i / \delta m$, in the same manner weight-average force affecting $\delta\tau$ is determined as

$\vec{F} = \sum_i m_i\vec{F}_i / \sum_i m_i = \sum_i m_i\vec{F}_i / \delta m$, whence $\vec{v}\delta m = \sum_i m_i\vec{v}_i$,

$\vec{F}\delta m = \sum_i m_i\vec{F}_i$, ρ – density $\rho = \delta m / \delta\tau$, $\delta m = \rho\delta\tau = \sum_i m_i$.

Theorem on system of mass points impulse change is applied to **elementary volume** of continuum $\delta\tau$, containing aggregation of particles $\sum_i m_i$, and not to final volume $\tau = \iiint_\tau \delta\tau$, strictly speaking, this is the essence of the **inductive** approach:

$$\frac{d}{dt}\sum_i m_i\vec{v}_i = \sum_i \vec{F}_i m_i + \oiint_{\sigma_{\delta\tau}} \sum_k \vec{\pi}_{nk}\sigma_k \,, \qquad (1)$$

where $\vec{\pi}_n\delta\sigma = \sum_k \vec{\pi}_{nk}\sigma_k$ - resultant surface force, affecting elementary surface $\delta\sigma = \sum_k \sigma_k$, at this $\delta\sigma \in \sigma_{\delta\tau}$, σ_k – site of surface $\delta\sigma$, which is occupied by particle $m_{\sigma k}$, that is under action of strain $\vec{\pi}_{nk}$, $\sigma_{\delta\tau} = \oiint_{\sigma_{\delta\tau}} \delta\sigma$.

Formula (1) in equivalent representation takes the above-used form

$$\frac{d}{dt}(\vec{v}\,\delta m) = \vec{F}\delta m + \oiint_{\sigma_{\delta\tau}} \vec{\pi}_n\delta\sigma \,, \qquad (2)$$

which results in, as was shown, continuum dynamics equation under strains.

But it is also possible to make it in a more simple way, without applying theorem on integral mean value, and using passage to the limit, $\delta\tau = \delta x\delta y\delta z \to 0$, if, elementary volume $\delta\tau = \delta x\delta y\delta z$ is taken as parallelepiped with faces, which are parallel to coordinate planes, where differences are equal $\delta x = x_2 - x_1, \delta y = y_2 - y_1, \delta z = z_2 - z_1$. Impulse change law for parallelepiped with sum of particles $\sum_i m_i = \delta m = \rho\delta\tau$ shall be represented with account to bulk and surface forces affecting $\delta\tau = \delta x\delta y\delta z$. In the result, in application

to parallelepiped, theorem on impulse change will take the form which is similar to (2):

$$\frac{d}{dt}(\vec{v}\,\delta m) = \vec{F}\delta m + (\vec{\pi}_{x1} + \vec{\pi}_{-x1})\delta y \delta z + \qquad (3)$$

$$+ (\vec{\pi}_{y2} + \vec{\pi}_{-y1})\delta x \delta z + (\vec{\pi}_{z2} + \vec{\pi}_{-z1})\delta x \delta y \ ,$$

where, just as previously,

$$\vec{v}\,\delta m = \sum_i m_i \vec{v}_i , \vec{F}\delta m = \sum_i m_i \vec{F}_i , \vec{\pi}_n \delta\sigma = \sum_k \vec{\pi}_{nk}\sigma_k$$

Comparison of (3) with (2) shows that surface integral for parallelepiped is calculated by its faces

$$\sigma_{\delta\tau} = (\delta y \delta z)_1 \bigcup (\delta y \delta z)_2 \bigcup (\delta x \delta z)_1 \bigcup (\delta x \delta z)_2 \bigcup (\delta x \delta y)_1 \bigcup (\delta x \delta z)_2$$

In the result, the following value of integral is obtained

$$\oiint_{\sigma_{\delta\tau}} \sum_k \vec{\pi}_{nk}\sigma_k = (\vec{\pi}_{x2} + \vec{\pi}_{-x1})\delta y \delta z + (\vec{\pi}_{y2} + \vec{\pi}_{-y1})\delta x \delta z +$$

$$+ (\vec{\pi}_{z2} + \vec{\pi}_{-z1})\delta x \delta y$$

Using formula

$$\frac{d\delta\tau}{dt} = div\vec{v}\,\delta\tau$$

and equality of strains

$$\vec{\pi}_{-x1} = -\vec{\pi}_{x1}, \vec{\pi}_{-y1} = -\vec{\pi}_{y1}, \vec{\pi}_{-z1} = -\vec{\pi}_{z1}, \qquad (4)$$

having divided by $\delta\tau = \delta x \delta y \delta z$, we derive formula

$$\rho\frac{d\vec{v}}{dt} + \vec{v}\left(\frac{d\rho}{dt} + \rho div\vec{v}\right) = \rho\vec{F} + (\vec{\pi}_{x2} - \vec{\pi}_{x1})/\delta x +$$

$$+ (\vec{\pi}_{y2} - \vec{\pi}_{y1})/\delta y + (\vec{\pi}_{z2} - \vec{\pi}_{z1})/\delta z \qquad (5)$$

It is obvious that passage to the limit in (5) $\delta x \to 0, \delta y \to 0,$ $\delta z \to 0,$ gives continuum dynamics equation under strains

$$\rho\frac{d\vec{v}}{dt}+\vec{v}(\frac{d\rho}{dt}+\rho div\vec{v})=\rho\vec{F}+\frac{\partial\vec{\pi}_x}{\partial x}+\frac{\partial\vec{\pi}_y}{\partial y}+\frac{\partial\vec{\pi}_z}{\partial z} \qquad (6)$$

When continuity equation is equal to zero (no sources and drains)

$$\frac{d\rho}{dt}+\rho div\vec{v}=0$$

this equation (6) exactly matches with equation (10) from **§3**.

Theorem of *Ostrogradsky-Gauss* and theorem on integral average value are not used here.

Energy balance equation is derived in a similar manner. For elementary parallelepiped $\delta\tau=\delta x\delta y\delta z$ of continuum, energy conservation law is formulated as follows:

$$\begin{aligned}\frac{d}{dt}[(E+|\vec{v}|^2/2)\delta m]=(\vec{F}\delta m,\vec{v})+[(\vec{\pi}_{x2},\vec{v}_{(x_2)})+(\vec{\pi}_{-x1},\vec{v}_{(x_1)})]\delta y\delta z+\\ +[(\vec{\pi}_{y2},\vec{v}_{(y_2)})+(\vec{\pi}_{-y1},\vec{v}_{(y_1)})]\delta z\delta x+\\ +[(\vec{\pi}_{z2},\vec{v}_{(z_2)})+(\vec{\pi}_{-z1},\vec{v}_{(z_1)})]\delta x\delta y-\\ -\{(q_{(x_2)}-q_{(x_1)})\delta y\delta z+(q_{(y_2)}-q_{(y_1)})\delta z\delta x+\\ +(q_{(z)2}-q_{(z)1})\delta x\delta y\}+Q\delta m\end{aligned} \qquad (7)$$

where $\vec{v}_{(x_m)}, m=1,2$ - values of velocity vector on areas $\delta y\delta z$ in sections x_1, x_2 etc., $\vec{q}=q_{(x)}\vec{i}+q_{(y)}\vec{j}+q_{(z)}\vec{k}$ - heat flow vector, $(E+|\vec{v}|^2/2)\delta m$ - total energy of volume $\delta\tau$, $(\vec{F}\delta m,\vec{v})$ - bulk force power,

$$\begin{aligned}[(\vec{\pi}_{x2},\vec{v}_{(x_2)})+(\vec{\pi}_{-x1},\vec{v}_{(x_1)})]\delta y\delta z+\\ +[(\vec{\pi}_{y2},\vec{v}_{(y_2)})+(\vec{\pi}_{-y1},\vec{v}_{(y_1)})]\delta z\delta x+\\ +[(\vec{\pi}_{z2},\vec{v}_{(z_2)})+(\vec{\pi}_{-z1},\vec{v}_{(z_1)})]\delta x\delta y\end{aligned}$$

- sum of powers of surface forces affecting pairs of faces
- $\delta y\delta z, \delta z\delta x, \delta x\delta y$ of parallelipiped,

$$-\{(q_{(x_2)}-q_{(x_1)})\delta y\delta z+(q_{(y_2)}-q_{(y_1)})\delta z\delta x+(q_{(z_2)}-q_{(z_1)})\delta x\delta y\}$$

- heat flows through these faces, $Q\delta m -$ energy source or power consumer in volume $\delta\tau$.

Having divided both parts (7) by $\delta\tau=\delta x\delta y\delta z$ and subtending parallelepiped to point $\delta x\to 0,\delta y\to 0,\delta z\to 0,$ we derive energy balance equation in continuous medium:

$$\rho\frac{d}{dt}(E+|\vec{v}|^2/2)+(E+|\vec{v}|^2/2)(\frac{d\rho}{dt}+\rho div\vec{v})= \quad (8)$$

$$=\rho(\vec{F},\vec{v})+\frac{\partial}{\partial x}(\vec{\pi}_x,\vec{v})+\frac{\partial}{\partial y}(\vec{\pi}_y,\vec{v})+\frac{\partial}{\partial z}(\vec{\pi}_z,\vec{v})-div\vec{q}+\rho Q,$$

from which, in accordance with basic *Fourier* conduction law $\vec{q}=-\lambda gradT$ and $dE=c_v dT$, for *Newton* strain nonsymmetric tensor /8/ obtained is heat influx equation

$$\rho c_v\frac{dT}{dt}=div(\lambda gradT)+\mu\sum_{i=1}^{3}\sum_{j=1}^{3}(\frac{\partial v_i}{\partial x_j})^2-pdiv\vec{v}-(\frac{1}{3}\mu-\mu')(div\vec{v})^2$$

(Heat energy crossing area $\delta\sigma$ in unit time is equal to $(\vec{q},\vec{n})\delta\sigma$).

Given above inductive approach possesses direct link with fundamental laws of physics. It is necessary to note that continuity equation (6) was derived in /1/ from law of conservation of matter with the help of *inductive* approach. According to law of conservation of matter, mass $\delta m=\rho\delta\tau=\sum_i m_i$ of individual volume $\delta\tau$ is a constant δm =*const,* due to which

$$\frac{d\delta m}{dt}=0,\frac{d(\rho\delta\tau)}{dt}=0,\rho\frac{d(\delta\tau)}{dt}+\delta\tau\frac{d\rho}{dt}=0 \quad (9)$$

Due to $\frac{d\delta\tau}{dt}=\delta\tau\cdot div\vec{v}$ (9) results in

$$\rho\delta\tau div\vec{v}+\delta\tau\frac{d\rho}{dt}=0,$$

having reduced $\delta\tau$, we come to continuity equation

$$\frac{d\rho}{dt}+\rho div\vec{v}=0$$

The purpose of the above-given two paragraphs was: to show disadvantages of using deductive method and accuracy of inductive approach.

The point is that in deriving false symmetry of continuum strain tensor, *deductive method* is traditionally used.

§5. Paradoxes of the first *Helmholtz* theorem

It is known /1/ that *Helmholtz*, excluding pressure p from the *Navier-Stokes* incompressible fluid equations /1/, obtained equations for velocity rotation

$$rot\vec{v}=(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z})\vec{i}+(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x})\vec{j}+(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y})\vec{k}$$

Bearing in mind that dimensionality $rot\vec{v}$ coinsides with dimensionality of angular velocity $\vec{\omega}$, *Helmholtz* tried to connect them with the help of *velocity formula for solid body points /*1*/:*

$$\vec{v}=\vec{v}_0+[\vec{\omega},(\vec{r}-\vec{r}_0)], \qquad (1)$$

where $\vec{v}_O, \vec{r}_O$ velocity and pole radius vector in relation to which body instant rotation occurs at the moment. Effort to express components of angular velocity $\vec{\omega}=\omega_x\vec{i}+\omega_y\vec{j}+\omega_z\vec{k}$ via components of linear velocity $\vec{v}=u\vec{i}+v\vec{j}+w\vec{k}$ led *Helmholtz* to solving in relation to $\omega_x,\omega_y,\omega_z$ in linear algebraic system

$$\begin{aligned} u&=u_0+\omega_y(z-z_0)-\omega_z(y-y_0),\\ v&=v_0+\omega_z(x-x_0)-\omega_x(z-z_0),\\ w&=w_0+\omega_x(y-y_0)-\omega_y(x-x_0), \end{aligned} \qquad (2)$$

determinant of which is equal to zero, therefore the given system with assigned

$$\delta u=u-u_0,\delta v=v-v_0,\delta w=w-w_0,$$
$$\delta x=x-x_0,\delta y=y-y_0,\delta z=z-z_0$$

has infinite number of solution sets, which first of all results from equality

$$\left| \vec{v} - \vec{v}_0 \right| = \left| \left[\vec{\omega}, (\vec{r} - \vec{r}_0)\right] \right| = |\vec{\omega}| |\vec{r} - \vec{r}_0| \sin\alpha ;$$

system consistency results from orthogonality condition $\vec{v} - \vec{v}_0$ to $\vec{r} - \vec{r}_0$: $(\vec{v}, \vec{r} - \vec{r}_0)$=0. Indeed, if ω_x is selected as reference arbitrary variable, solution set for system (2) is represented in the following form

$$\omega_z = \frac{\delta v}{\delta x} + \omega_x \frac{\delta z}{\delta x}, \omega_y = \omega_x \frac{\delta y}{\delta x} - \frac{\delta w}{\delta x} \qquad (3)$$

Helmholtz (*Loytzyansky* in/1/?) acted absolutely differently. He differentiates (2) by *x,y,z,* assuming that $\vec{v}_0$, $\vec{r}_0$, $\vec{\omega}$, are constant and as the result he obtained the following formulae for components of *solid body* angular velocity:

$$\omega_x = -\frac{\partial v}{\partial z}, \omega_y = -\frac{\partial w}{\partial x}, \omega_z = -\frac{\partial u}{\partial y},$$

$$\omega_x = \frac{\partial w}{\partial y}, \omega_y = \frac{\partial u}{\partial z}, \omega_z = \frac{\partial v}{\partial x}, \qquad (4)$$

Paradoxical, but use of method (4) *Helmholtz* (*Loytsyansky*?) to the solution of linear equations system

$$\sum_{i=1}^{N} a_{ij} x_j = b_i, i = 1,\dots,N$$

gives values of desired quantities in the form $x_j = \frac{\partial b_i}{\partial a_{ij}}, j = 1,\dots,N, i = 1,\dots,N$, i.e. obtained are ***N*N*** of values of desired unknowns $\boldsymbol{x_j}$ despite of prescribed ***N.*** It is obvious that they may form any combinations of the following type

$$x_j = \sum_m (\alpha_m \frac{\partial b_m}{\partial a_{mj}}) \Big/ \sum_m \alpha_m , j = 1,\dots,N,$$

where α_m - arbitrary numbers. In this case, in systems (4), this fact may be used as follows.

Multiplying upper line by n – lower line by m, adding and dividing them by $m+n \neq 0$, will find values

$$\omega_x = \frac{m}{m+n}\frac{\partial w}{\partial y} - \frac{n}{m+n}\frac{\partial v}{\partial z},$$

$$\omega_y = \frac{m}{m+n}\frac{\partial u}{\partial z} - \frac{n}{m+n}\frac{\partial w}{\partial x}, \qquad (5)$$

$$\omega_z = \frac{m}{m+n}\frac{\partial v}{\partial x} - \frac{n}{m+n}\frac{\partial u}{\partial y}$$

Helmholtz (*Loytsyansky*?) from this infinite diversity of angular velocity components used only one aggregation resulted from (5) with m=n=1:

$$(rot\vec{v})_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 2\omega_x, (rot\vec{v})_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 2\omega_y,$$

$$(rot\vec{v})_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\omega_z, \qquad (6)$$

where known expression of velocity rotation was used.

It is obvious that with no values of m, n, expressions (5) $\omega_x, \omega_y, \omega_z$ ***do not coinside*** with solutions (3) for system (2). Even if to take as value of unrestricted variable

$$\omega_x = \frac{1}{2}(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}),$$

substituting it into found solutions (3) for system (2)

$$\omega_z = \frac{\delta v}{\delta x} + \omega_x\frac{\delta z}{\delta x} = \frac{\delta v}{\delta x} + \frac{1}{2}(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})\frac{\delta z}{\delta x} \neq \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}),$$

$$\omega_y = \omega_x\frac{\delta y}{\delta x} - \frac{\delta w}{\delta x} = \frac{1}{2}(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})\frac{\delta y}{\delta x} - \frac{\delta w}{\delta x} \neq \frac{1}{2}(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})$$

we'll receive evidence that ***Helmholtz*** formulae (6) ***) are not solutions of system*** (2), QED. ***Paradoxical fact,*** using incorrect solution (6) *Helmholtz* instead of formula (1) refers to expression

$$\vec{v} = \vec{v}_0 + [\frac{1}{2}rot\vec{v}, (\vec{r} - \vec{r}_0)], \qquad (7)$$

which has nothing in common with formula of solid body velocity (1), since (6) is not the solution for system (2).

Due to the fact that *Helmholtz* formulae (6) are not solutions for system (2), this expression is far from being equivalent to formula (1) of solid body particle velocity! (This formula is erroneous, it is not suitable for calculation of solid body particles velocity.) However, this expression became later a prototype for *Helmholtz* first theorem, on which we will focus in more details.

Helmholtz first theorem is derived from approximation formula, presenting members with first derivatives of *Taylor* series in the following form

$$\vec{v}(\vec{r}+\delta\vec{r},t)=\vec{v}(\vec{r},t)+\bar{S}\delta\vec{r} \tag{8}$$

Helmholtz, by analogy with incorrect expression (7) transformed (8) to equivalent form (refer /1/)

$$\vec{v}(\vec{r}+\delta\vec{r},t)=\vec{v}(\vec{r},t)+\frac{1}{2}[rot\vec{v},\delta\vec{r}]+\dot{S}\delta\vec{r}\,, \tag{9}$$

having introduced strain rates tensor $\dot{S}$. Formula (9) is the content of the *Helmholtz* first theorem. Comparing (9) with incorrect expression (7), *Helmholtz* declares $\dot{S}\ \delta\vec{r}$ to be a ***deformation displacement.*** If to bear in mind that expression $\vec{v}\ =\vec{v}_0+[\frac{1}{2}rot\vec{v},(\vec{r}-\vec{r}_0)]$ separately does not have any physical sense at all, as opposed to solid body point velocity $\vec{v}\ =\vec{v}_0+[\vec{\omega},(\vec{r}-\vec{r}_0)]$, then formulated notion of deformation displacement is very doubtful from the physical point of view.

More contensive is ***displacement*** $\bar{S}\delta\vec{r}$ in (8) and because of results (7), (9), (10) §2, this product may be declared a ***deformation displacement.*** *Let's show that Taylor series* (8) *may be assigned with infinite number of forms containing* $rot\vec{v}$ *, thus, it would be proved that there is the infinite number of "deformation displacements" of Helmholtz type* $\dot{S}\ \delta\vec{r}$. With this purpose we'll represent *Taylor* series (8) in projections on coordinate axis:

$$u(x+\delta x,y+\delta y,z+\delta z,t)=u(x,y,z,t)+\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\frac{\partial u}{\partial z}\delta z,$$

$$v(x+\delta x, y+\delta y, z+\delta z, t) = v(x,y,z,t) + \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \frac{\partial v}{\partial z}\delta z, \quad (10)$$

$$w(x+\delta x, y+\delta y, z+\delta z, t) = w(x,y,z,t) + \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z$$

Equivalent transformation of *Taylor* series (10) has the following form:

$$u(x+\delta x, y+\delta y, z+\delta z, t) = u(x,y,z,t) + \frac{\partial u}{\partial x}\delta x - \frac{b-1}{b}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\delta y +$$

$$+\frac{b-1}{b}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\delta z + \left(\frac{b-1}{b}\frac{\partial v}{\partial x} + \frac{1}{b}\frac{\partial u}{\partial y}\right)\delta y + \left(\frac{b-1}{b}\frac{\partial w}{\partial x} + \frac{1}{b}\frac{\partial u}{\partial z}\right)\delta z,$$

$$v(x+\delta x, y+\delta y, z+\delta z, t) = v(x,y,z,t) + \frac{b-1}{b}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\delta x + \frac{\partial v}{\partial y}\delta y -$$

$$-\frac{b-1}{b}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\delta z + \left(\frac{b-1}{b}\frac{\partial u}{\partial y} + \frac{1}{b}\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{b-1}{b}\frac{\partial w}{\partial y} + \frac{1}{b}\frac{\partial v}{\partial z}\right)\delta z,$$

$$w(x+\delta x, y+\delta y, z+\delta z, t) = w(x,y,z,t) - \frac{b-1}{b}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\delta x +$$

$$+\frac{b-1}{b}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\delta y + \frac{\partial w}{\partial z}\delta z + \left(\frac{b-1}{b}\frac{\partial u}{\partial z} + \frac{1}{b}\frac{\partial w}{\partial x}\right)\delta x + \left(\frac{b-1}{b}\frac{\partial v}{\partial z} + \frac{1}{b}\frac{\partial w}{\partial y}\right)\delta y,$$

By analogy with presentation of *Helmholtz* (9) *Taylor* series (11) may be represented in vector form:

$$\vec{v}(\vec{r}+\delta\vec{r}, t) = \vec{v}(\vec{r}, t) + \frac{b-1}{b}[rot\vec{v}, \delta\vec{r}] + S_b\delta\vec{r} \quad , \qquad (12)$$

where in $S_b\delta\vec{r}$ (product of line S_b by column $\delta\vec{r}$) there is a matrix

$$S_b = \begin{pmatrix} \frac{\partial u}{\partial x}, \left(\frac{b-1}{b}\frac{\partial v}{\partial x} + \frac{1}{b}\frac{\partial u}{\partial y}\right), \left(\frac{b-1}{b}\frac{\partial w}{\partial x} + \frac{1}{b}\frac{\partial u}{\partial z}\right) \\ \left(\frac{b-1}{b}\frac{\partial u}{\partial y} + \frac{1}{b}\frac{\partial v}{\partial x}\right), \frac{\partial v}{\partial y}, \left(\frac{b-1}{b}\frac{\partial w}{\partial y} + \frac{1}{b}\frac{\partial v}{\partial z}\right) \\ \left(\frac{b-1}{b}\frac{\partial u}{\partial z} + \frac{1}{b}\frac{\partial w}{\partial x}\right), \left(\frac{b-1}{b}\frac{\partial v}{\partial z} + \frac{1}{b}\frac{\partial w}{\partial y}\right), \frac{\partial w}{\partial z} \end{pmatrix}, \delta\vec{r} = \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}\begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} \quad , (13)$$

b=const, $b \neq 0, |b| < \infty$. When *b=2* from (12) the first theorem of *Helmholtz* (9) is derived, since $S_2 = \dot{S}$, i.e. tensor S_2 is equal to strain rates tensor, with *b=1 universal* formula (12) transfers into *Taylor* series in initial representation of (8), since $S_1 = \overline{S}$, i.e. S_1 is equal to motion matrix. Everywhere above, velocity rotation has components

$$(rot\vec{v})_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, (rot\vec{v})_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, (rot\vec{v})_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (14)$$

Therefore, expansion into *Taylor* series (10) may be given another equivalent form:

$$u(x+\delta x, y+\delta y, z+\delta z, t) = u(x,y,z,t) + \frac{\partial u}{\partial x}\delta x -$$

$$-(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\delta y + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})\delta z + \frac{\partial v}{\partial x}\delta y + \frac{\partial w}{\partial x}\delta z,$$

$$v(x+\delta x, y+\delta y, z+\delta z, t) = v(x,y,z,t) + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\delta x +$$

$$+\frac{\partial v}{\partial y}\delta y - (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})\delta z + \frac{\partial u}{\partial y}\delta x + \frac{\partial w}{\partial y}\delta z,$$

$$w(x+\delta x, y+\delta y, z+\delta z, t) = w(x,y,z,t) - (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})\delta x +$$

$$+(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})\delta y + \frac{\partial w}{\partial z}\delta z + \frac{\partial u}{\partial z}\delta x + \frac{\partial v}{\partial z}\delta y \quad (15)$$

By analogy with (12) projections (15) are reduced into expression

$$\vec{v}(\vec{r} + \delta\vec{r}, t) = \vec{v}(\vec{r}, t) + [rot\vec{v}, \delta\vec{r}] + \delta\vec{r}\,\overline{S} \quad (16)$$

where $\overline{S}$ - is nonsymmetric displacement matrix from §1, $\delta\vec{r}\,\overline{S}$ contain products of lines by columns. *Helmholtz* called the following formula a vorticity

$$\vec{\omega} = 1/2 rot\vec{v}, \quad (17)$$

basing on false presentation (7), which was discussed above.

As is known /1/, /4/, velocity rotation in hydrodynamics is connected, first of all, with ***velocity circulation***:

$$(rot\vec{v},\vec{n}) = \lim_{\sigma\to 0}\frac{\oint\limits_{C}(\vec{v},d\vec{r})}{\sigma}, \qquad (18)$$

and components $rot\vec{v}$ in the given in (14) form are derived in particular from this determination (18), therefore formula $\vec{\Omega} = rot\vec{v}$ must be used for vorticity. If to proceed from determination of *Helmholtz,* deformation displacements are $S_b\delta\vec{r}$, contained in universal formula (12).

§6. Paradoxes of *Stokes* hypothesis

1[0]. *Stokes* hypothesis was based on wrong *Helmholtz* formula

$$\vec{v} = \vec{v}_0 + [1/2rot\vec{v},(\vec{r}-\vec{r}_0)]$$

§5 describes this fallacious *Helmholtz* formula, on the basis of which extension into *Taylor* series is formulated in the form of *Helmholtz first theorem /1/:*

$$\vec{v}(\vec{r}+\delta\vec{r},t) = \vec{v}(\vec{r},t) + 1/2[rot\vec{v},\delta\vec{r}] + \dot{S}\delta\vec{r}$$

Further on, Stokes, having admitted in this *Taylor* series the summand $\dot{S}\,\delta\vec{r}$ as deformation displacement occurring under action of surface forces, *theoretically* assumed that tangential stress are proportional to ***doubled values*** of components of strain rates tensor $\dot{S}$:

$$\pi_{ij(c)} = 2\mu\dot{S}_{ij}, i \neq j\ ,$$

thanks to which, tangential stress happened to be symmetrical due to symmetry of matrix $\dot{S}$. *Stokes neglected the equivalent member of Taylor* series (**this is the major error of his hypothesis**):

$$1/2\,[rot\vec{v},\delta\vec{r}] \equiv \dot{S}\delta\vec{r}\ .$$

His hypothesis *Stokes* (another name - generalized *Newton* law /1/), formulated for flow of viscous fluids and gases in tensor form

$$\pi_c = -(p + 2/3\mu div\vec{v})E + 2\mu\dot{S}, \qquad (1)$$

In index notations components of strains tensor π_c are represented

in a shorter form

$$\pi_{ii(c)} = -(p + 2/3\mu div\vec{v}) + 2\mu\frac{\partial v_i}{\partial x_i}, i = 1,2,3,$$

$$\pi_{ji(c)} = \mu(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}), i \neq j, \qquad (2)$$

and symmetry of strain tensor in the form

$$\pi_{ij(c)} = \pi_{ji(c)},\ i \neq j, i, j = 1,2,3 \qquad (3)$$

Noticed is the circumstance that diagonal $\dot{S}$ tensor elements enter normal strains with ***doubled value***, which is connected with slanting of tangential stress (2) into *Newton* friction law $\pi_{ns} = \mu\frac{\partial v_s}{\partial n}$.

For instance, *Newton* friction law with $v_s = u, n = y, s = x$ gives formula $\pi_{yx(н)} = \mu\frac{\partial u}{\partial y}$, and according to *Stokes* hypothesis (2) it comes out that $\pi_{yx(c)} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$. Apparent mismatch is connected with inclusion of paradoxical excess member $\mu\frac{\partial v}{\partial x}$. (By the way, inconsistence of *Stokes* hypothesis was indicated by *L.D.Landau*.)

Here arises question about adequacy of member $2\mu\frac{\partial v_i}{\partial x_i}$ in normal strain in the sense that selection of coefficient “2” is justified only by Stokes hypothesis (1), according to which the following artificially created equality must be realized

$$\pi_{ji(c)} = 2\dot{S}_{ji}, i \neq j \qquad (4)$$

In the preceeding paragraph § 5 for *Taylor* series

$$\vec{v}(\vec{r} + \delta\vec{r}, t) = \vec{v}(\vec{r}, t) + \bar{S}\delta\vec{r}$$

was given an infinite number of equivalent formulations, containing velocity rotation,

$$\vec{v}(\vec{r}+\delta\vec{r},t)=\vec{v}(\vec{r},t)+\frac{b-1}{b}[rot\vec{v},\delta\vec{r}]+S_b\delta\vec{r} \tag{5}$$

If to follow logic of *Stokes* hypothesis, strains selected on the basis of considerations of their proportionality to components of tensor S_b , will be defined in the following form

$$\pi_{xx}=-(p+b/3\mu div\vec{v})+b\mu\frac{\partial u}{\partial x},\pi_{yy}=-(p+b/3\mu div\vec{v})+b\mu\frac{\partial v}{\partial y},$$

$$\pi_{zz}=-(p+b/3\mu div\vec{v})+b\mu\frac{\partial w}{\partial z},$$

$$\pi_{yx}=\mu(\frac{b-1}{b}\frac{\partial v}{\partial x}+\frac{1}{b}\frac{\partial u}{\partial y}),\pi_{yz}=\mu(\frac{b-1}{b}\frac{\partial v}{\partial z}+\frac{1}{b}\frac{\partial w}{\partial y}),$$

$$\pi_{xy}=\mu(\frac{b-1}{b}\frac{\partial u}{\partial y}+\frac{1}{b}\frac{\partial v}{\partial x}),\pi_{xz}=\mu(\frac{b-1}{b}\frac{\partial u}{\partial z}+\frac{1}{b}\frac{\partial w}{\partial x}),$$

$$\pi_{zx}=\mu(\frac{b-1}{b}\frac{\partial w}{\partial x}+\frac{1}{b}\frac{\partial u}{\partial z}),\pi_{zy}=\mu(\frac{b-1}{b}\frac{\partial w}{\partial y}+\frac{1}{b}\frac{\partial v}{\partial z})$$

For $b\neq 2$ tangential stresses are *nonsymmetric*, symmetry occurs onle with ***b*=2**, when $S_2=\dot{S}$. It is also obvious that when *b=1* matrix (6) transforms into displacement matrix $S_1=\bar{S}$.

There is the infinite number of viscous flows (only some of them are given here), where symmetry of *Stokes* strains (3) contradicts to fundamental *Newton* friction law $\pi_{ns}=\mu\frac{\partial v_s}{\partial n}$, where $\vec{n}$ crosswise to $\vec{s}$ direciton, $v_s=(\vec{v},\vec{s})$ - projection of velocity on ort direction $\vec{s}$, π_{ns} - tangential stress.

2^0. Paradox of *Stokes* hypothesis $S_2=\dot{S}$ in *Poiseuille* and *Couette* flows

In laminar flow between parallel tight and rigid channel walls (*Poiseuille* flows), velocities are defined as solution of equation

$$\frac{dp}{dx} = \mu \frac{d^2u}{dy^2}, \quad u(\pm b) = 0,$$

in the form of dependences

$$u = -\frac{1}{2\mu} \cdot \frac{dp}{dx}(b^2 - y^2), (\frac{dp}{dx} = const), v \equiv 0,$$

(*here* $v_1 = u, v_2 = v, v_3 = w, x_1 = x, x_2 = y, x_3 = z$).

In this flow, longitudinal tangential stress towards $\boldsymbol{x}$ by formula (2) is equal to $\pi_{yx(c)} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = \frac{dp}{dx} y$. Due to the fact that there is no flow along crosswise direction $v \equiv 0$, which means that derivative is equal to zero $\frac{\partial v}{\partial x} = 0$, according to *Newton* friction law crosswise tamgential stress towards $\boldsymbol{y}$ will also be equal to zero: $\pi_{xy(н)} = \mu \frac{\partial v}{\partial x} = 0$, which can be fully explained by absence of *cross flow* towards *y*, since $v \equiv 0$ along the whole channel. According to *Stokes* hypothesis about symmetry of tangential stress $\pi_{yx(c)} = \pi_{xy(c)}$, there is a paradox, which is in the fact that crosswise tangential stress is not equal to zero, because $\pi_{xy(c)} = \pi_{yx(c)} = \frac{dp}{dx} y \neq 0$, which conflicts with fundamental *Newton* friction law, according to which, as was proved above, crosswise tamgential stress is equal to zero $\pi_{xy(н)} = 0$.

Thus, in *Poiseuille* flow crosswise and longitudinal tangential stress are not equal to each other, i.e. their symmetry in relation to left principal diagonal of strain tensor π_c in the form (2) does not take place. This contradiction is quite similar to stated below paradox with tangential stress by Stokes (2) in *Hagen-Poiseuille* flow in pipe. For *Couette* flow, nonsymmetry of tangential stress is defined similarly.

3^0. Paradox of *Stokes* hypothesis $S_2 = \dot{S}$ in arbitrary flows

For infinite number of flows symmetric tangential stress in *Stokes* hypothesis (2) has zero values, i.e. in all points of the flow are equal to zero

$$\pi_{ji(c)} = \mu(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) \equiv 0, i \neq j, i = 1,2,3, j = 1,2,3$$

Let's restrict ourselves with giving a short list of flows with velocity components, in which this fact take place

1) $u=F(\sin k_1 x \cos k_1 y)$, $v=F(-\cos k_1 x \sin k_1 y)$,

2) $u= U(\sin k_2 x \cos k_2 y - \cos k_2 x \sin k_2 y)$,

$v= U(\sin k_2 x \cos k_2 y - \cos k_2 x \sin k_2 y)$,

3) $u=W(-\cos k_3 x \sin k_3 y)$, $v=W(\sin k_3 x \cos k_3 y)$,

4) $u=Q(\sin k_4 x \sin k_4 y)$, $v=Q(\cos k_4 x \cos k_4 y)$,

5) $u=T(\sin k_5 x \sin k_5 y)$, $v=T(\cos k_5 x \cos k_5 y)$,

6) $u_x=M(\sin k_6 x \sin k_6 y + \cos k_6 y \cos k_6 x)$,

$v=M(\sin k_6 x \sin k_6 y + \cos k_6 y \cos k_6 x)$,

7) $u =S(e^{k_7(x+y)})$, $v=-S(e^{k_7(x+y)})$,

for three-dimensional flows: 8) $u=D((e^{k_8 y} - e^{k_8 z})e^{k_8 x})$,

$v=D((e^{k_8 z} - e^{k_8 x})\ e^{k_8 y})$, $w=D((e^{k_8 x} - e^{k_8 y})e^{k_8 z})$,

where coefficients $k_i = const, i = 1,2,3,4,5,6,7,8,$ are selected arbitrarily from infinite interval $-\infty < k_i < +\infty$. Standing here differentiable functions *F,U,W,Q,T,M,S,D* are also arbitrary in choice. It is obvious that new any linear combinations of type *u=F+U, v=F+U* etc. may be formed from the indicated list. Velocity fields 1-7 correspond to plane flows and satisfy two-dimensional continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, velocity fields 8) satisfy three-

minesional continuity equation $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$.

For all these flows, *Stokes* tangential stress is identically equal to zero in all flow points:

$$\pi_{ji(c)}=\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\equiv 0, i\neq j, i=1,2,3, j=1,2,3,$$

$$\pi_{ij(c)}=\pi_{ji(c)}=0, i\neq j$$

For compressible gas, number of flows where zero strains of *Stokes* take place $\pi_{ijc}=\pi_{jic}=0, i\neq j$,

infinitely increases due to presence of variable density ρ in continuity equation. Thus, in flows with components of velocity of 1,2,3,4,5,6,7,8 type, symmetrical strains (2) reduce to zero, $\pi_{ij(c)}=0$, $i\neq j$, and it comes so, that viscous fluid motion occurs without friction, which contradicts to fundamental *Newton* law $\pi_{ns}=\mu\partial v_s/\partial n$. It is easy to calculate by this formula that tangential stress in the indicated flows is not equal to zero

$$\pi_{ji(н)}=\mu\frac{\partial v_i}{\partial x_j}\neq 0, \quad i\neq j \tag{6}$$

4⁰. Paradox of *Stokes* hypothesis $S_2=\dot{S}$ in *Hagen–Poiseuille* flow in a round pipe

Viscous fluid laminar flow in a round pipe (ref./1/) in cylindrical coordinates has the velocities

$$V_z=-\frac{1}{4\mu}\frac{dp}{dz}(a^2-r^2), V_r=0, V_\varphi=0, \ \frac{dp}{dz}=const,$$

where a – pipe radius, z - axial, r - radial coordinates. According to *Stokes* hypothesis (2) symmetrical tangential stress are equal to each other and are calculated by formula /1/

$$\pi_{zr(c)}=\pi_{rz(c)}=\mu\left(\frac{\partial V_r}{\partial z}+\frac{\partial V_z}{\partial r}\right)=\frac{1}{2}\frac{dp}{dz}r \tag{7}$$

Let’s consider flow in positive direction towards axis z, it appears when pressure drops $\dfrac{dp}{dz} = const < 0$.

At this, by formula (7) longitudinal tangential stress are negative $\pi_{rz(c)} < 0$, by *Stokes* hypothesis (although flow in crosswise direction ***r*** is absent) due to symmetry (3) exist and are negative crosswise tangential stress

$$\pi_{zr(c)} = \pi_{rz(c)}, \;\; \pi_{zr(c)} < 0$$

Their directions are shown in Fig.1.

$\Leftarrow -z$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $+z \Rightarrow$

pipe symmetry axis	0 $\qquad\qquad\qquad\qquad$ z
$\pi_{zr(н)} = 0$	$\rightarrow\rightarrow\rightarrow v_z$
$\dfrac{dp}{dz} < 0 \rightarrow\rightarrow\rightarrow v_z$	$\leftarrow \pi_{rz(c)} < 0$
$\leftarrow \pi_{rz(н)} < 0$	$\uparrow \pi_{zr(c)} < 0$
pipe wall	$+a$

$+r \Downarrow$ $\qquad\qquad$ Fig.1

According to fundamental *Newton* friction law, crosswise tangential stress are equal to zero $\pi_{zr(н)} = \mu \partial v_r / \partial z = 0$, because $v_r = 0$, longitudinal tangential stress are negative

$$\pi_{rz(н)} = \mu \partial v_z / \partial r = 1/2(dp/dz)r < 0,$$

and both - tangential stress by *Stokes* hypothesis as well as tangential stress by *Newton* formula $\pi_{rz(c)} < 0, \pi_{rz(н)} < 0$ have equal directions up the stream. With positive gradient $dp/dz > 0$ fluid in

pipe flows in negative direction of z axis (right to left), tangential stress change signs due to (7)

$$\pi_{rz(c)} > 0, \pi_{zr(c)} > 0$$

According to *Newton* friction law

$$\pi_{rz(н)} > 0, \pi_{zr(н)} = \mu \frac{\partial v_r}{\partial z} = 0$$

and flow will be directed "$-z$", which is displayed on Fig.2.

180^o turn of Fig.2 results in Fig.3, which displays that flow in pipe is directed in the same direction as on Fig.1.

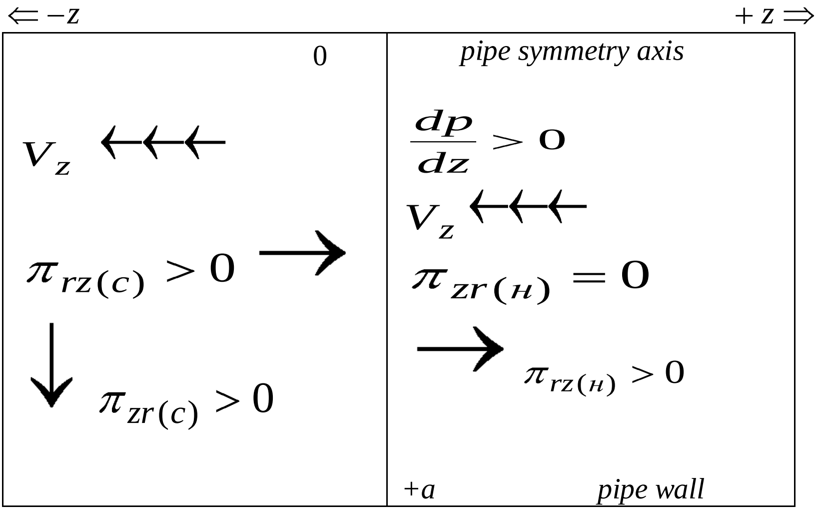

Fig.2

Contradiction is in location of nonzero tangential stress by *Stokes* hypothesis $\pi_{zr(c)}$, which in Fig. 3 ***is directed towards pipe wall***, and in Fig.1 ***is directed to pipe axis with equal flow direction, whereas strain*** $\pi_{rz(c)}$ ***in both cases is directed up the stream!*** On the contrary, according with *Newton* law tangential stress is equal to zero:

$$\pi_{zr(н)} = \mu \frac{\partial v_r}{\partial z} = 0,$$

since $v_r = 0$ because of absence of radial flow.

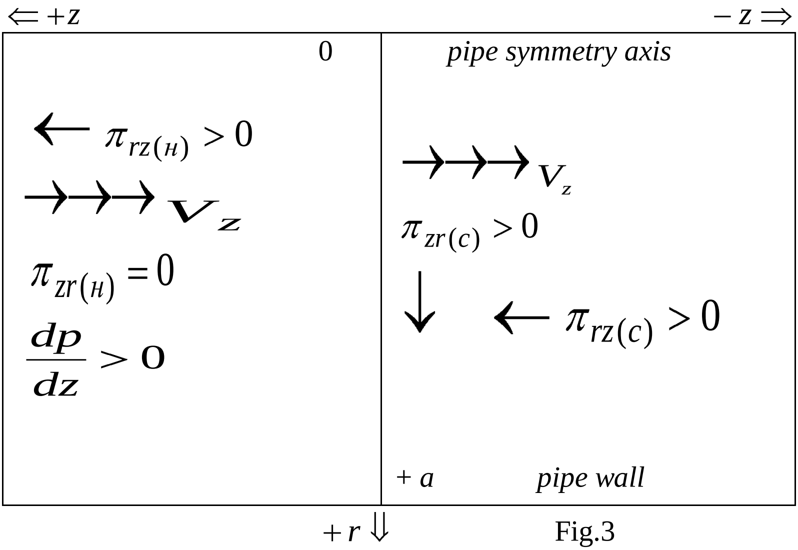

Fig.3

It is obvious that for nonsymmetric tensor (6) contradiction confirmed by Fig.1-3, does not arise, while for symmetric *Stokes* strain tensor (2) there is an indicated in Fig. 1,2,3 paradox. The same paradox with directions of symmetric *Stokes* tangential stress, obviously obtained in *Poiseuille* and *Couette* flows.

§7. Nonsymmetric *Newton* strains tensor. Paradoxes of determination of viscous normal stress are related to hypotheses on pressure and to *Pascal* law for ideal liquids

Law of linear dependence of strains on strain rates proposed by *Stokes* in **1845** as inadequate generalization of *Newton* friction law, can be related to *Helmholtz* formula or Taylor series, which is the same:

$$\vec{v}(\vec{r}+\delta\vec{r},t) = \vec{v}(\vec{r},t) + 1/2[rot\vec{v},\delta\vec{r}] + \dot{S}\delta\vec{r}\,, \qquad (1)$$

equivalent representation (1): $\delta\vec{v} = \tilde{S}\delta\vec{r} + \dot{S}\delta\vec{r}$.

This formula needs to be presented detailed form

$$\delta v_i = \sum_{j=1}^{3}[\frac{1}{2}(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i})+\frac{1}{2}(\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i})]\delta x_j, i=1,2,3 \quad (2)$$

Referring (2) to infinitesimal period δt, we obtain ***convectional acceleration component***

$$\frac{\delta v_i}{\delta t} = \sum_{j=1}^{3}[\frac{1}{2}(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i})+\frac{1}{2}(\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i})]\frac{\delta x_j}{\delta t}, i=1,2,3$$

On the basis of (2) *Stokes* suggested a hypothesis that strains are proportional to doubled *deformation displacement* $2\dot{S}\delta\vec{r}$. The hypothesis was formulated in the form of *generalized Newton law* with symmetrical strains tensor

$$\pi_{ij(c)} = -(p+\frac{2}{3}\mu div\vec{v})\delta_{ij}+\mu(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}), i, j=1,2,3, \quad (3)$$

where δ_{ij} - *Kronecker* symbol. As was marked, appearance of multiplier “2” in $2\dot{S}\ \delta\vec{r}$ is connected with *fitting* tangential stress by *Stokes* (3) $\pi_{ji(c)} = \mu(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}), i \neq j$ to *Newton* friction law (6) **§6**: $\pi_{ji(н)} = \mu\frac{\partial v_i}{\partial x_j},\ i \neq j$. **In this connection there arises a question: if strains (3) are caused by deformation displacements** $\sum_{j=1}^{3}\frac{1}{2}(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i})\delta x_j$, **then the same strains cause rotation displacements in (2)** $\frac{1}{2}[rot\vec{v},\delta\vec{r}]_i \Rightarrow \sum_{j=1}^{3}\frac{1}{2}(\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i})\delta x_j$?

Especially so because, rotational motion includes the same gradients $\frac{\partial v_i}{\partial x_j}, \frac{\partial v_j}{\partial x_i}$, that are also in $\pi_{ji(c)},\ i \neq j$, moreover $\sum_{j=1}^{3}\frac{1}{2}(\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i})\frac{\delta x_j}{\delta t}$ is the constituent of *convectional*

acceleration!

Paradoxical neglect of *Stokes* of these forces was motivated by fitting of the hypothesis (3) to *erroneous statement of symmetry of strains tensor*. (Nonsymmetry of continuum strains tensor will be rigorously proven in the following paragraphs.)

Answer to the question arisen above is the insistent *need* of considering strains

$$\pi^{*}_{ji} = \frac{1}{2}\mu\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right),$$

Corresponding to *rotational displacements of medium*, equally with strains

$$\pi^{**}_{ji} = \frac{1}{2}\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

arising in flow due to *deformation displacements*, *i.e. it is impossible to neglect such important component of general motion as* $\frac{1}{2}[rot\vec{v}, \delta\vec{r}] \equiv \mathfrak{S}\delta\vec{r}$. In the result of this, total force is determined in the following form

$$\pi_{ji(н)} = -(p + \frac{1}{3}\mu div\vec{v})\delta_{ji} + \pi^{*}_{ji} + \pi^{**}_{ji},$$

that after substitution $\pi^{*}_{ij}, \pi^{**}_{ij}$ leads to nonsymmetrical strains tensor

$$\pi_{ji(н)} = -(p + \frac{1}{3}\mu div\vec{v})\delta_{ij} + \mu\frac{\partial v_i}{\partial x_j}, i, j = 1,2,3 \qquad (4)$$

This formula automatically corresponds to both – *Taylor* series $\delta v_j = \sum_{i=1}^{3}\frac{\partial v_j}{\partial x_i}\delta x_i$ as well as *Newton* friction law $\pi_{ns} = \mu\frac{\partial v_s}{\partial n}$ and gives *nonsymmetric* values of tangential stress (6) **§6**: $\pi_{ji(н)} = \mu\frac{\partial v_i}{\partial x_j}, i \neq j$. In Cartesian coordinates, it will look as follows:

$$\pi_{yx(н)} = \mu\frac{\partial u}{\partial y}, \qquad \pi_{xy(н)} = \mu\frac{\partial v}{\partial x}, \qquad \pi_{xz(н)} = \mu\frac{\partial w}{\partial x},$$

$$\pi_{zx(н)} = \mu \frac{\partial u}{\partial z}, \quad \pi_{zy(н)} = \mu \frac{\partial v}{\partial z}, \quad \pi_{yz(н)} = \mu \frac{\partial w}{\partial y}$$

Normal strains, due to equalities (7) §2 , are equal

$$\bar{S}_{xx} = \dot{e}_x, \bar{S}_{yy} = \dot{e}_y, \bar{S}_{zz} = \dot{e}_z,$$

$$\dot{e}_x = \partial u / \partial x, \dot{e}_y = \partial v / \partial y, \dot{e}_z = \partial w / \partial z$$

$$\pi_{xx(н)} = -(p + \frac{1}{3}\mu div\vec{v}) + \mu \frac{\partial u}{\partial x}, \ \pi_{yy(н)} = -(p + \frac{1}{3}\mu div\vec{v}) + \mu \frac{\partial v}{\partial y},$$

$$\pi_{zz(н)} = -(p + \frac{1}{3}\mu div\vec{v}) + \mu \frac{\partial w}{\partial z},$$

where, as opposed to normal strains of *Stokes* hypothesis, there is $1/3\mu div\vec{v}$, (effort to justify this formula was made by *A.V.Lykov* in /8/).

Paradoxes of defining viscous normal stress

1st hypothesis on pressure states: arithmetical mean value of viscous normal stresses must be equal to pressure (refer /1/):

$$\frac{\pi_{xx(н)} + \pi_{yy(н)} + \pi_{zz(н)}}{3} = -p \qquad (5)$$

Due to this hypothesis, into normal stresses in *Stokes* law (3) *artificially* was included member $2/3\mu div\vec{v}$ and into (4) by analogy included was member $1/3\mu div\vec{v}$. These artificial hypothetical inclusions provide for unconditional execution of pressure hypothesis (5).

It is known that hypothesis (5) for ideal fluids $\vec{p}_n = -p\vec{n}$ is *Pascal* law, unconditionally executed for *incompressible viscous fluids* $div\vec{v} = 0$, even when normal stresses in them are defined with arbitrary coefficient $\bar{\mu}$:

$$\pi_{ii} = -p + \bar{\mu} \frac{\partial v_i}{\partial x_i}, \ i = 1,2,3, \qquad (6)$$

and tangential stress are defined by *Newton* law $\pi_{ji} = \mu \frac{\partial v_i}{\partial x_j}, i \neq j$.

When $\bar{\mu}$=0 in (6), $\pi_{ii} = -p, \;\; i = 1,2,3$, result in *Pascal* law.

When value of $\bar{\mu} = \mu$ normal and tangential stress are determined by a single representation

$$\pi_{ji} = -p\delta_{ij} + \mu\frac{\partial v_i}{\partial x_j}, i, j = 1,2,3, \qquad (7)$$

i.e. there is canonical nonsymmetric tensor

$$\pi = -pE + \mu\bar{S} \qquad (8)$$

If during definition of viscous normal strains, to proceed from 1st hypothesis (5), there is an arbitrariness related to selection of coefficient $\bar{\mu}$. (quite possible that in simulation of turbulent flows this circumstance may play an important role)

2st hypothesis on pressure states: that normal stresses in *compressible gas* $div\vec{v} \neq 0$ must satisfy equation

$$\frac{\pi_{xx(н)} + \pi_{yy(н)} + \pi_{zz(н)}}{3} = -p + \mu' div\vec{v}, \qquad (9)$$

where arbitrary coefficient μ' (for its selection there are no any physical justifications) is declared a coefficient of *bulk viscosity* or *second viscosity coefficient*

It is not difficult to calculate that 2st hypothesis is observed for normal strains in the following form

$$\pi_{ii(н)} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v}] + \mu\frac{\partial v_i}{\partial x_i}, i = 1,2,3, \quad (10)$$

i.e. nonsymmetric strains tensor takes the following form

$$\pi_{ji(н)} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v}]\delta_{ij} + \mu\frac{\partial v_i}{\partial x_j}, i, j = 1,2,3, \quad (11)$$

$$\pi_{н} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v})]E + \mu\bar{S} \qquad (12)$$

As already said, there is a definite arbitrariness in selection of multiplier μ'. Disposing this arbitrariness, let's assume $\mu' = \frac{1}{3}\mu$, thanks to which normal strains (10) in *compressible gases* $div\vec{v} \neq 0$ *will take the following form*

$$\pi_{ii} = -p + \mu \frac{\partial v_i}{\partial x_i}, \quad i = 1,2,3, \qquad (13)$$

coinsiding with normal strains in incompressible $div\vec{v} = 0$ fluids (7) $\bar{\mu} = \mu$, i.e. (12) will transform into (8) and there will be ***a uniform rheological law*** for both – incompressible fluids, as well as for compressible gases:

$$\pi = -pE + \mu\bar{S}\ , \pi_{ji} = -p\delta_{ij} + \mu \frac{\partial v_i}{\partial x_j}, i, j = 1,2,3 \quad (14)$$

Only hypothesis (9) will take other form

$$\frac{\pi_{xx(н)} + \pi_{yy(н)} + \pi_{zz(н)}}{3} = -p + \frac{1}{3}\mu div\vec{v}\,, \qquad (15)$$

realized for normal strains (13)!

In the event of incompressible fluids with $div\vec{v} = 0$ hypothesis (15) automatically transforms into *Pascal* law.

Thus it was shown: 1) that inclusion of complexes $1/3\mu div\vec{v}$ or $2/3\mu div\vec{v}$ is some absolutely obvious slanting of viscous normal strains to pressure hypothesis (5); 2) formulae of normal strains depend on form of pressure hypothesis.

Refusal of artificial additions of types of $2/3\mu div\vec{v}$, $1/3\mu div\vec{v}$, $(2/3\mu - \mu')div\vec{v}, (1/3\mu - \mu')div\vec{v}$, considerably simplifies strains tensor, and, accordingly, vicous compressible gas dynamics equiations with $div\vec{v} \neq 0$.

Described above paradoxes with pressure hypotheses and *Newton* rheological law (14) allow making a conclusion: proportionality of viscous part of normal strains to diagonal elements of displacement tensor $\bar{S}$, is justified by *Newton* friction law, due to which tangential stress are proportional to nondiagonal elements of tensor $\bar{S}$.

Newton rheological law (14) releases equations of *viscous compressible gases* dynamics in dissipative part from mixed derivatives, in connection with this, three-dimensional equations are reduced on **12 derivatives**, and if to compare with equations of *Navier-Stokes*, then in total sum in new equations **18 derivatives** less contained in comparison with equations of *Navier-Stokes*. Besides,

presence of mixed derivatives affects type of equations, i.e. for quasi-parabolicity creates specific difficulties in researching stability of difference schemes for their numerical solution.

In §5 it was noted that in universal formula

$$\vec{v}(\vec{r}+\delta\vec{r},t)=\vec{v}(\vec{r},t)+\frac{b-1}{b}[rot\vec{v},\delta\vec{r}]+S_b\delta\vec{r}$$

when *b=1*, there is matrix $S_1=\bar{S}$, therefore transfer to displacement tensor $\bar{S}$ is performed when *b=1*

$$\vec{v}(\vec{r}+\delta\vec{r},t)=\vec{v}(\vec{r},t)+\bar{S}\delta\vec{r} \; ,$$

thus, it is proved that for ***formation of Newton strains tensor there is no special need in the first Helmholtz theorem, it is necessary to proceed directly from the Taylor series*** $\vec{v}(\vec{r}+\delta\vec{r},t)=\vec{v}(\vec{r},t)+\bar{S}\delta\vec{r}$.

Division of *Taylor* series into symmetric $\dot{S}\ \delta\vec{r}$ and antisymmetric $1/2[rot\vec{v},\delta\vec{r}]$ parts is directly connected with formation of symmetric continuum strains tensor and deformations theory (this issue is considered here in Chapter 9).

L. D. Landau, E. M. Lifshitz in /10/ laid down the following conditions as additional requirement to rheological law of continuum: "*… in viscous fluid rotating about axis like a solid body with uniform angular velocity all tangential stress must reduce to zero*". Strains (6) meet this requirement (refer /8/). Indeed, in cylindric system $r,\ \varepsilon,\ z$ /1/, *Newton* strains (7) have the following forms:

$$\pi_{r\varepsilon}=\mu(\frac{\partial v_\varepsilon}{\partial r}-\frac{v_\varepsilon}{r}),\ \ \pi_{\varepsilon r}=\frac{\mu}{r}\frac{\partial v_r}{\partial\varepsilon},\ \ \pi_{zr}=\mu\frac{\partial v_r}{\partial z},$$

$$\pi_{rz}=\mu\frac{\partial v_z}{\partial r},\pi_{z\varepsilon}=\mu\frac{\partial v_\varepsilon}{\partial z},\ \ \pi_{\varepsilon z}=\frac{\mu}{r}\frac{\partial v_z}{\partial\varepsilon}.$$

$$\pi_{rr}=-[p+(\frac{1}{3}\mu-\mu')div\vec{v}]+\mu\frac{\partial v_r}{\partial r},$$

$$\pi_{\varepsilon\varepsilon}=-[p+(\frac{1}{3}\mu-\mu')div\vec{v}]+\mu(\frac{1}{r}\frac{\partial v_\varepsilon}{\partial\varepsilon}+\frac{v_r}{r}),$$

$$\pi_{zz} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v}] + \mu\frac{\partial v_z}{\partial z},$$

$$div\vec{v} \equiv \frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{1}{r}\frac{\partial v_\varepsilon}{\partial \varepsilon},$$

for fluid $div\vec{v} = 0$, rotating with uniform angular velocity ω, velocities are equal (α - angle between $\vec{\omega}$ and $\vec{r}$):

$$v_r = 0, v_z = 0, v_\varepsilon = \|[\vec{\omega}, \vec{r}]\| =$$
$$= |\vec{\omega}||\vec{r}|\sin\alpha = \omega r\sin\alpha,$$

Accordingly, all tangential stress are equal to zero

$$\pi_{rz} = 0, \pi_{zr} = 0, \pi_{r\varepsilon} = 0, \pi_{\varepsilon r} = 0, \pi_{z\varepsilon} = 0, \pi_{\varepsilon z} = 0.$$

§8. Backhground for *fallacious* conclusion on strain tensor symmetry

Let's consider the system of material points with masses m_i, $i=1,2,...,N$ and with radius-vectors r_i, $i=1,2,...,N$, upon which influence forces $\vec{F}_i, i = 1,...,N$. Resultant (major) force is equal to $\vec{F}^* = \sum_i \vec{F}_i$.

Moment of resultant (major) force is equal to $\vec{M}_c = [\vec{r}_c, \vec{F}^*]$, where $\vec{r}_c = \sum_i m_i\vec{r}_i / \sum_i m_i$ radius vector of center of mass, denoted further simply $\vec{r}_c = \vec{r}$; and resultant (major) moment is equal

$$\vec{M} = \sum_i [\vec{r}_i, \vec{F}_i].$$

Theorem 1. *For the system of material points moment of resultant force is not equal in general case to resultant moment:* $\vec{M}_c \neq \vec{M}$.

Indeed, there is the obvious inequality

$$[\sum_i m_i\vec{r}_i / \sum_i m_i, \sum_i \vec{F}_i] \neq \sum_i [\vec{r}_i, \vec{F}_i],$$

besides, the simpliest example, confirming the theorem, is the pair of

opposite forces $\vec{F}_1 = -\vec{F}_2$, resultant force here is equal to zero:

$$\vec{F}^* = \vec{F}_1 + \vec{F}_2 = 0, therefore\ \vec{M}_c = 0,$$

whereas resultant (major) moment of couple is not equal to zero:

$$\vec{M} = [\vec{r}_1, \vec{F}_1] + [\vec{r}_2, \vec{F}_2] = [\vec{r}_1 - \vec{r}_2, \vec{F}_1] \neq 0, (\vec{r}_1 - \vec{r}_2) \# \vec{F}_1 \quad (1)$$

Theorem 2. *For mutually given perpendicular vectors* $\vec{Q} \perp \vec{F}$ *there exists infinite number of vectors* $\vec{r}^* = x^* \vec{i} + y^* \vec{j} + z^* \vec{k}$, *for which there is an equality* $[\vec{r}^*, \vec{F}] = \vec{Q}$. *If mutually given perpendicular vectors* $\vec{Q} \perp \vec{r}^*$, *there is an infinite number of vectors* $\vec{F}^*$, *for which this equality exists.*

Actually, this equation, solved in relation to x^*, y^*, z^*, has determinant equal to zero

$$y^* F_z - z^* F_y = Q_x,\ z^* F_x - x^* F_z = Q_y,\ x^* F_y - y^* F_x = Q_z, (2)$$

where

$$\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}, \vec{Q} = Q_x \vec{i} + Q_y \vec{j} + Q_z \vec{k}$$

From the equality $(\vec{r}^*, \vec{Q}) = 0$, comes expression $x^* Q_x + y^* Q_y + z^* Q_z = 0$, which is represented in the following form

$$y^* Q_y + z^* Q_z = -x^* Q_x$$

and is solved along with the first equation of this system (2). In the result obtained is the infinite set of solutions

$$y^* = \frac{Q_x(Q_z - x^* F_y)}{F_y Q_y + F_z Q_z}, \quad z^* = \frac{-Q_x(x^* F_z + Q_y)}{F_y Q_y + F_z Q_z}.$$

The second part of the theorem is proved in the same way.

Theorem 3. *For the given vector* $\vec{Q}$ *it is possible to find infinite number of vectors* $\vec{r}^*$ *and* $\vec{F}$, *for which there is equality* $[\vec{r}^*, \vec{F}] = \vec{Q}$.

In this theorem system (2) is solved together with equations

$$(\vec{r}^*, \vec{Q}) = 0 \text{ and } (\vec{F}, \vec{Q}) = 0\ ,$$

in the result, for 6 given values $x^*, y^*, z^*, F_x, F_y, F_z$ obtained are 5 equations, i.e. 5 unknowns will be expressed via the sixth, for instance as in theorem 2, via x^*.

Resultant force is in the right part of dynamics equation (10) §3, where with the purpose shortness of strain will be denoted by inferior indexes

$$\vec{\pi}_x = \vec{\pi}_1, \vec{\pi}_y = \vec{\pi}_2, \vec{\pi}_z = \vec{\pi}_3,$$

Multiplying (10) §3 vectorially by $\vec{r}$, we'll find moments

$$[\vec{r}, \rho\frac{d\vec{v}}{dt}] = [\vec{r}, \rho\vec{F} + \sum_{j=1}^{3}\frac{\partial\vec{\pi}_j}{\partial x_j}] \qquad (3)$$

It is obvious that in (3) ***moment of major force*** is equal to

$$\vec{M}_c = [\vec{r}, \rho\vec{F} + \sum_{j=1}^{3}\frac{\partial\vec{\pi}_j}{\partial x_j}] \qquad (4)$$

Firstly, symmetry of strains tensor in /1/ is derived from theorem on moment of momentum change (angular momentum), represented in the form of integral

$$\frac{d}{dt}\iiint_\tau[\vec{r}, \vec{v}\rho\delta\tau] = \iiint_\tau[\vec{r}, \vec{F}\rho\delta\tau] + \oint\!\!\oint_\sigma[\vec{r}, \vec{\pi}_n\delta],$$

with introduction of differentiation in the left part for integral

$$\iiint_\tau\frac{d}{dt}[\vec{r}, \vec{v}\rho\delta\tau] = \iiint_\tau[\vec{r}, \vec{F}\rho\delta\tau] + \oint\!\!\oint_\sigma[\vec{r}, \vec{\pi}_n\delta], \qquad (5)$$

which is unjustified, if volume $\tau = \tau(t)$ depends on time. As for the left part of this expression, due to formula (7) §1, there is inequality

$$\frac{d}{dt}\iiint_\tau[\vec{r}, \vec{v}\rho\delta\tau] = \iiint_\tau\frac{\partial}{\partial t}[\vec{r}, \vec{v}\rho\delta\tau] + \oint\!\!\oint_\sigma[\vec{r}, \vec{v}\rho](\vec{v}, \vec{n})\delta\sigma \neq \iiint_\tau\frac{d}{dt}[\vec{r}, \vec{v}\rho\delta\tau]$$

From thus formulated theorem in /1/, in the same way, in other textbooks, derived was wrong in general case relation

$$[\vec{r}, \rho\frac{d\vec{v}}{dt}] = [\vec{r}, \rho\vec{F}] + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}[\vec{r}, \vec{\pi}_j] \qquad (6)$$

Mismatch of expression (6) to fundamental theorem on change of moment of momentum of system of particles is considered below in **§9.** In the right part of (6) there is a resultant (major) moment

$$\vec{M} = [\vec{r}, \rho\vec{F}] + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}[\vec{r}, \vec{\pi}_j] \,, \tag{7}$$

since (7) is derived from relation

$$\frac{d}{dt}\iiint_{\tau} [\vec{r}, \vec{v}\rho\delta\tau] = \iiint_{\tau}[\vec{r}, \rho\vec{F}]\delta\tau + \oint\!\!\oint_{\sigma}[\vec{r}, \sum_{j=1}^{3}\vec{\pi}_j \cos(\vec{n}\text{€}x_j)]\delta\sigma \,, \tag{8}$$

where in the right part, there is a summary (in the form of triple integral) major moment of bulk forces $\iiint_{\tau}[\vec{r}, \rho\vec{F}]\delta\tau$ and major moment of surface forces $\oint\!\!\oint_{\sigma}[\vec{r}, \sum_{j=1}^{3}\vec{\pi}_j \cos(\vec{n}\text{€}x_j)]\delta\sigma$, obviously, having different radius-vectors $\vec{r}$. In fact, from (8) instead of formula (6), according to (9) §1

$$\frac{d}{dt}\iiint_{\tau}\Phi\delta\tau = \iiint_{\tau}(\frac{\partial\Phi}{\partial t} + div(\Phi\vec{v}))\delta\tau + \iiint_{\tau}\Phi\frac{\partial\delta\tau}{\partial t} \,,$$

derived is exact expression

$$\frac{\partial}{\partial t}[\vec{r}, \vec{v}\rho] + div([\vec{r}, \vec{v}\rho]u)\vec{i} + div([\vec{r}, \vec{v}\rho]v)\vec{j} +$$
$$+ div([\vec{r}, \vec{v}\rho]w)\vec{k} + \frac{[\vec{r}, \vec{v}\rho]}{\delta\tau}\frac{\partial\delta\tau}{\partial t} = [\vec{r}, \rho\vec{F}] + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}[\vec{r}, \vec{\pi}_j], \tag{9}$$

consequently, left part of (6) is not equal to left part of (9), and from (9) in no way results fact of symmetry of strain tensor in the general case.

By differentiation (7) in /1/ is represented as follows

$$\vec{M} = [\vec{r}, \rho\vec{F} + \sum_{j=1}^{3}\frac{\partial\vec{\pi}_j}{\partial x_j}] + \sum_{j=1}^{3}[\frac{\partial\vec{r}}{\partial x_j}, \vec{\pi}_j], \tag{10}$$

Due to (4) *major moment* (10) is represented in the following form

$$\vec{M} = \vec{M}_c + \sum_{j=1}^{3}[\frac{\partial \vec{r}}{\partial x_j}, \vec{\pi}_j] , \qquad (11)$$

By ***theorem 1*** these moments are not equal to each other: $\vec{M}_c \neq \vec{M}$.

In /1/ and other textbooks equality of these moments is traditionally assumed, which is possible only when the last member in (11) is equal to zero:

$$\sum_{j=1}^{3}[\frac{\partial \vec{r}}{\partial x_j}, \vec{\pi}_j] = 0 \qquad (12)$$

For the moment of resultant forces $\vec{M}_c$ be equal to resultant moment $\vec{M}$ ***condition*** (12) ***must be observed in each point of continuum.***

Theorem 4. *Condition* (12) *is observed in the following three situations*:

1. *Strains are parallel to coordinate axes:*

$$\vec{\pi}_j \parallel \frac{\partial \vec{r}}{\partial x_j}, j = 1,2,3$$

2. *Strain components are symmetrical:*

$$\pi_{ij} = \pi_{ji}, i, j = 1,2,3. \qquad (13)$$

Thus, symmetry of strain tensor (13) and resulting from it symmetry of strains contradict to ***theorem 1***, which is confirmed by given in §6 examples 1-8 and list of paradoxes. **This implyies logical conclusion that *symmetry of strain tensor* by *Stokes* hypothesis *is not physical property of continuum*, but is the consequence of *artificial* equating of major moment $\vec{M}$ to the moment of major force $\vec{M}_c$, i.e. *mathematical condition* of equality of these moments in the sense that if in some point of flow conditions of (13) are observed simultaneously, the major moment of force becomes equal to the major moment $\vec{M}_C = \vec{M}$, if these conditions (13) are not observed, they are not equal to each other: $\vec{M}_c \neq \vec{M}$.** Consequently, theorem on change

of moment of momentum was incorrectly applied to arbitrary volume of continuum, which was stated in §3, because this equating occurred due to equating left parts of expressions (3) and (6). Strain tensor is not symmetric in the general case, nonsymmetry for individual flows was indicated in /3/. The following conclusion is just: ***for viscous media, Newton rheological law with nonsymmtric tangential stress §7 is just,*** since *Stokes* hypothesis leads to the above-mentioned contradictions.

§9. Momentum theorem does not mean symmetry of strain tensor. Strain tensor is nonsymmetric

For a system of material points with masses m_i, moving with velocities $\vec{v}_i$ and affected by forces $\vec{f}_i$, momentum thorem has the following form /5/:

$$\frac{d}{dt}\sum_{i=1}^{N}[\vec{r}_i, m_i\vec{v}_i] = \sum_{i=1}^{N}[\vec{r}_i, \vec{f}_i] \qquad (1)$$

For taken as a parallelepiped volume $\delta\tau = \delta x\delta y\delta z$ of individual particles of continuum, radius-vector $\vec{r}_m$ moves towards the point of application of resultant bulk force $\vec{F}\delta m$, accordingly, radius-vectors of resultant surface forces are directed towards points of their action on each face of parallelipiped:

$$\vec{r}_{(x2)} \to \vec{\pi}_{x2},\ \vec{r}_{(x1)} \to \vec{\pi}_{-x1},\ \vec{r}_{(y2)} \to \vec{\pi}_{y2},\ \vec{r}_{(y1)} \to \vec{\pi}_{-y1},$$

$$\vec{r}_{(z2)} \to \vec{\pi}_{z2},\ \vec{r}_{(z1)} \to \vec{\pi}_{-z1}.$$

Due to this, according to fundamental theorem (1) for $\delta\tau = \delta x\delta y\delta z$ ***major moment of forces*** will be presented in the following form

$$\begin{aligned}\vec{M}^* = [\vec{r}_m, \delta m\vec{F}] + [\vec{r}_{(x2)}, \vec{\pi}_{x2}\delta y\delta z] + [\vec{r}_{(x1)}, \vec{\pi}_{-x1}\delta y\delta z] + \\ + [\vec{r}_{(y2)}, \vec{\pi}_{y2}\delta z\delta x] + [\vec{r}_{(y1)}, \vec{\pi}_{-y1}\delta z\delta x] + \\ + [\vec{r}_{(z2)}, \vec{\pi}_{z2}\delta x\delta y] + [\vec{r}_{(z1)}, \vec{\pi}_{-z1}\delta x\delta y]\end{aligned} \qquad (2)$$

And resultant force is in the right part (3) §4 and is equal to

$$\vec{F}_{рез} = \vec{F}\delta m + (\vec{\pi}_{x_2} + \vec{\pi}_{-x1})\delta y\delta z + (\vec{\pi}_{y2} + \vec{\pi}_{-y1})\delta x\delta z +$$
$$+ (\vec{\pi}_{z2} + \vec{\pi}_{-z1})\delta x\delta y$$

It is obvious that by ***theorem 1*** **major moment of force *is not equal to* major moment of forces:** $[\vec{r}, \vec{F}_{рез}] \neq \vec{M}^*$**.** Let's consider in more details the integrand in the left part (8) **§8** $\frac{d}{dt}\iiint_\tau [\vec{r}, \vec{v}\rho\delta\tau]$, equal to $[\vec{r}, \delta m\vec{v}]$ and representing moment of momentum $\delta m\vec{v}$ of mass $\delta m = \rho\delta\tau$, where $\delta\tau$ is individual volume containing mass $\delta m = \sum_i m_i$, at this each particle has velocity $\vec{v}_i$, i.e. there is a conformity $\vec{r}_i \to m_i \to \vec{v}_i, i = 1,2,...,$ due to which the major moment of system of particles vectors in volume $\delta\tau$ will be equal to $\sum_i [\vec{r}_i, m_i\vec{v}_i]$. According to ***theorem 1*** it is not equal to moment of major momentum $[\vec{r}, \delta m\vec{v}]$:

$$\sum_i [\vec{r}_i, m_i\vec{v}_i] \neq [\vec{r}, \delta m\vec{v}],$$

because

$$[\vec{r}, \delta m\vec{v}] = \left[\frac{\sum_i m_i\vec{r}_i}{\sum_i m_i}, \sum_i m_i \frac{\sum_i m_i\vec{v}_i}{\sum_i m_i}\right],$$

accordingly, equality between them is possible only when conditions of ***theorems 2 or 3*** are observed**.** In passage to the limit $\vec{r}$ in the considered expression, $[\vec{r}, \delta m\vec{v}]$ is a specific radius-vector of continuum point, meaning that by ***theorem 3*** , $\frac{d\vec{v}}{dt}$ will be the arbitrary vector, therefore equality between $\iiint_\tau [\vec{r}, \rho\frac{d\vec{v}}{dt}]\delta\tau$, which

was obtained by integration in the total volume of continuum for both parts (3) §8

$$\iiint_\tau [\vec r, \rho \frac{d\vec v}{dt}]\delta\tau = \iiint_\tau [\vec r, \rho\vec F + \sum_{j=1}^{3} \frac{\partial \vec\pi_j}{\partial x_j}]\delta\tau \,,$$

and $\iiint_\tau \frac{d}{dt}[\vec r, \vec v\rho\delta\tau]$ is impossible, thus the following inequality is justified

$$\frac{d}{dt}\iiint_\tau [\vec r, \vec v\rho\delta\tau] \neq \iiint_\tau \frac{d}{dt}[\vec r, \vec v\rho\delta\tau],$$

since by formula (7) **§1,** left part of this expression is equal to

$$\frac{d}{dt}\iiint_\tau [\vec r, \vec v\rho\delta\tau] = \iiint_\tau \frac{\partial}{\partial t}[\vec r, \vec v\rho\delta\tau] + \oiint_\sigma [\vec r, \vec v\rho](\vec v, \vec n)\delta\sigma \,,$$

and the right part transforms into expression

$$\iiint_\tau \frac{d}{dt}[\vec r, \vec v\delta m] = \iiint_\tau \{ [\frac{d\vec r}{dt}, \vec v\delta m] + [\vec r, \delta m\frac{d\vec v}{dt}] + [\vec r, \vec v\frac{d\delta m}{dt}] \} =$$

$$= \iiint_\tau \{ [\vec r, \delta m\frac{d\vec v}{dt}] + [\vec r, \vec v\frac{d\delta m}{dt}] \}$$

Absolutely similar conclusions need to be made for moment of bulk forces $[\vec r, \delta m\vec F]$, because $\vec F$ is the resultant of all forces $\vec F_i$ in volume $\delta\tau$, acting upon particles m_i of continuum, however

$$\delta m = \sum_i m_i \,,\ \vec F = \sum_i m_i\vec F_i / \sum_i m_i = \sum_i m_i\vec F_i \Big/ \delta m$$

By ***theorem 1***

$$[\vec r, \delta m\vec F] \neq \sum_i [\vec r_i, m_i\vec F_i],$$

therefore, for a specific vector $\vec r$ by ***theorem 3,*** equality

$$[\vec r, \delta m\vec F^*] = \sum_i [\vec r_i, m_i\vec F]$$

is possible only for such vector $\vec F^*$, which won't be in general case

equal to true $\vec{F}$: $\vec{F}^* \neq \vec{F}$, i.e. in the right part of (7) **§8** there is in fact $\vec{F}^*$, which is absolutely arbitrary vector. Classic formula from /1/

$$\frac{d}{dt}\iiint_{\tau}[\vec{r},\vec{v}\delta m] = \iiint_{\tau}[\vec{r},\vec{F}\delta m] + \oiint_{\sigma}[\vec{r},\vec{\pi}_n\delta\sigma] \qquad (3)$$

due to reasons given above does not correspond to fundamental theorem on change of moment of momentum (1).

Indeed, *moment of momentum* theorem

$$\frac{d}{dt}\sum_{i=1}^{N}[\vec{r}_i, m_i\vec{v}_i] = \sum_{i=1}^{N}[\vec{r}_i, \vec{f}_i]$$

must be represented in individual volume $\delta\tau$ in the following form

$$\frac{d}{dt}\sum_{i}[\vec{r}_i, m_i\vec{v}_i] = \sum_{i}[\vec{r}_i, m_i\vec{F}_i] + \oiint_{\sigma_{\delta\tau}}\sum_{k}[\vec{r}_{\sigma k}, \vec{\pi}_{nk}\sigma_k],$$

accordingly, for volume τ in the form of triple integral

$$\iiint_{\tau}\frac{d}{dt}\sum_{i}[\vec{r}_i, m_i\vec{v}_i] = \iiint_{\tau}\sum_{i}[\vec{r}_i, m_i\vec{F}_i] + \iiint_{\tau}\oiint_{\sigma_{\delta\tau}}\sum_{k}[\vec{r}_{\sigma k}, \vec{\pi}_{nk}\sigma_k] \qquad (4)$$

According to theorems 2, 3, equality sign cannot be put between respective members of expressions (3) and (4), i.e. there are obvious inequalitites

$$\frac{d}{dt}\iiint_{\tau}[\vec{r},\vec{v}\delta m] \neq \iiint_{\tau}\frac{d}{dt}\sum_{i}[\vec{r}_i, m_i\vec{v}_i], \iiint_{\tau}[\vec{r},\vec{F}\delta m] \neq \iiint_{\tau}\sum_{i}[\vec{r}_i, m_i\vec{F}_i],$$

$$\oiint_{\sigma}[\vec{r},\vec{\pi}_n]\delta\sigma \neq \iiint_{\tau}\oiint_{\sigma_{\delta\tau}}\sum_{k}[\vec{r}_{\sigma k}, \vec{\pi}_{nk}\sigma_k], \qquad (5)$$

emphasizing difference of expressions (3) and (4). We need to take into account the circumstance that moving coordinates, having beginning in the center of mass $\delta m = \sum_i m_i$, accordingly, moving with acceleration $d\vec{v}/dt \neq 0$, will be *noninertial* system. Here, in (5), as previously $\vec{\pi}_n\delta\sigma = \sum_k \vec{\pi}_{nk}\sigma_k$ - is resultant surface force,

acting upon $\delta\sigma = \sum_k \sigma_k$, at this $\delta\sigma \in \sigma_{\delta\tau}$, σ_k – site of surface $\delta\sigma$, which is occupied by particle $m_{\sigma k}$, that is affected by strain $\vec{\pi}_{nk}$, vectors $\vec{r}_{\sigma k}$ are directed to particles $m_{\sigma k}$, laying on surface $\sigma_{\delta\tau}$, QED.

Symmetry of strain tensor does not result from (4), due to indicated above circumstances, i.e. $\pi_{ij} \neq \pi_{ji}, j \neq i,$, since in both parts (4) there are major moments, by ***theorem 1*** not equal to moments of resultant vecors in (3).

Let's once again show in detail fallacy of equality (6) **§8**, widely used in /1/,/2/,/3/,/4/ for proving of strain tensor symmetry. Let's represent it in the following form

$$[\vec{r}, \rho\frac{d\vec{v}}{dt}] - [\vec{r}, \rho\vec{F}] - \sum_{j=1}^{3} \frac{\partial}{\partial x_j}[\vec{r}, \vec{\pi}_j] = 0 \qquad (6)$$

Symmetry of continuum strain tensor is obtained theoretically from this expression and, as a result, slanting of symmetry of viscous fluid tangential stress in *Stokes* hypothesis /1/.

Let's prove that in fact left part of (6) is not equal to zero:

$$[\vec{r}, \rho\frac{d\vec{v}}{dt}] - [\vec{r}, \rho\vec{F}] - \sum_{j=1}^{3} \frac{\partial}{\partial x_j}[\vec{r}, \vec{\pi}_j] \neq 0$$

With this purpose we'll engage precise formulation of moment of momentum change theorem in elementary volume $\delta\tau$ of continuum:

$$\frac{d}{dt}\sum_i [\vec{r}_i, m_i\vec{v}_i] = \sum_i [\vec{r}_i, m_i\vec{F}_i] + \oiint_{\sigma_{\delta\tau}} \sum_k [\vec{r}_{\sigma k}, \vec{\pi}_{nk}\sigma_k], \qquad (7)$$

Between vectors $\vec{r}$ in (86) and $\vec{r}_i$, $\vec{r}_{\sigma k}$ in (7) there is an association

$$\vec{r}_i = \vec{r} + \vec{r}_i', \vec{r}_{\sigma k} = \vec{r} + \vec{r}_{\sigma k}',$$

wherefrom, an association between velocities results

$$\frac{d\vec{r}_i}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{r}_i'}{dt}, \qquad \vec{v}_i = \vec{v} + \vec{v}_i'$$

Substituting them into (7), we'll find

$$\frac{d}{dt}\sum_i [\vec{r} + \vec{r}_{i'}', m_i(\vec{v} + \vec{v}_i')] = \sum_i [\vec{r} + \vec{r}_i', m_i \vec{F}_i] +$$
$$+ \oiint_{\sigma_{\delta\tau}} \sum_k [\vec{r} + \vec{r}_{\sigma k}', \vec{\pi}_{nk}\sigma_k] \tag{8}$$

Let's make necessary transformations

$$\frac{d}{dt}\sum_i [\vec{r} + \vec{r}_{i'}', m_i(\vec{v} + \vec{v}_i')] = \frac{d}{dt}[\vec{r}, \sum_i m_i \vec{v}] + \frac{d}{dt}[\vec{r}, \sum_i m_i \vec{v}_i'] +$$
$$+ \frac{d}{dt}\sum_i [\vec{r}_i', m_i \vec{v}] + \frac{d}{dt}\sum_i [\vec{r}_i', m_i \vec{v}_i'] = \frac{d}{dt}[\vec{r}, \delta m \vec{v}] + \frac{d}{dt}[\vec{r}, \sum_i m_i \vec{v}_i'] +$$
$$+ \frac{d}{dt}\sum_i [\vec{r}_i', m_i \vec{v}] + \frac{d}{dt}\sum_i [\vec{r}_i', m_i \vec{v}_i']$$

Here, $\delta m = \sum_i m_i = \rho\delta\tau$ was taken into account in volume $\delta\tau$.

Having in mind $\frac{d\delta\tau}{dt} = \delta\tau div\vec{v}$, let's make the following transformations

$$\frac{d}{dt}[\vec{r}, \delta m \vec{v}] = [\frac{d\vec{r}}{dt}, \delta m \vec{v}] + [\vec{r}, \delta m \frac{d\vec{v}}{dt}] + [\vec{r}, \frac{d\delta m}{dt}\vec{v}] =$$
$$= [\frac{d\vec{r}}{dt}, \rho\delta\tau\vec{v}] + [\vec{r}, \rho\delta\tau \frac{d\vec{v}}{dt}] + [\vec{r}, \frac{d(\rho\delta\tau)}{dt}\vec{v}] = [\vec{r}, \rho\delta\tau \frac{d\vec{v}}{dt}] +$$
$$+ [\vec{r}, (\rho\frac{d\delta\tau}{dt} + \delta\tau\frac{d\rho}{dt})\vec{v}] = [\vec{r}, \rho\delta\tau \frac{d\vec{v}}{dt}] +$$
$$+ [\vec{r}, (\rho\delta\tau div\vec{v} + \delta\tau\frac{d\rho}{dt})\vec{v}] = [\vec{r}, \rho\delta\tau \frac{d\vec{v}}{dt}]$$

Here equality $[\frac{d\vec{r}}{dt}, \rho\delta\tau\vec{v}]$ =0 was used and also equality to zero of continuity equation: $\rho\delta\tau div\vec{v} + \delta\tau \frac{d\rho}{dt} = 0$.

Finally we have

$$\frac{d}{dt}\sum_i [\vec{r} + \vec{r}_{i'}, m_i(\vec{v} + \vec{v}_i')] = [\vec{r}, \rho\delta\tau \frac{d\vec{v}}{dt}] +$$

$$+\frac{d}{dt}[\vec{r}, \sum_i m_i\vec{v}_i'] + \frac{d}{dt}\sum_i [\vec{r}_i', m_i\vec{v}] + \frac{d}{dt}\sum_i [\vec{r}_i', m_i\vec{v}_i'] \qquad (9)$$

Further on, in the right part (8)

$$\sum_i [\vec{r} + \vec{r}_i', m_i\vec{F}_i] = [\vec{r}, \sum_i m_i\vec{F}_i] + \sum_i [\vec{r}_i', m_i\vec{F}_i] =$$

$$= [\vec{r}, \vec{F}\delta m] + \sum_i [\vec{r}_i', m_i\vec{F}_i] = [\vec{r}, \vec{F}\rho\delta\tau] + \sum_i [\vec{r}_i', m_i\vec{F}_i],$$

$$\oiint_{\sigma_{\delta\tau}} \sum_k [\vec{r} + \vec{r}_{\sigma k}', \vec{\pi}_{nk}\sigma_k] = \oiint_{\sigma_{\delta\tau}} \sum_k [\vec{r}, \vec{\pi}_{nk}\sigma_k] +$$

$$+ \oiint_{\sigma_{\delta\tau}} \sum_k [\vec{r}_{\sigma k}', \vec{\pi}_{nk}\sigma_k],$$

where by theorem of *Ostrovsky-Gauss* and theorem on mean for elementary volume $\delta\tau$ there is

$$\oiint_{\sigma_{\delta\tau}} \sum_k [\vec{r}, \vec{\pi}_{nk}\sigma_k] = \oiint_{\sigma_{\delta\tau}} [\vec{r}, \sum_k \vec{\pi}_{nk}\sigma_k] = \oiint_{\sigma_{\delta\tau}} [\vec{r}, \vec{\pi}_n\delta\sigma] =$$

$$= \iiint_{\delta\tau} \sum_j \frac{\partial[\vec{r}, \vec{\pi}_j]}{\partial x_j}\delta\tau' = \sum_j \frac{\partial[\vec{r}, \vec{\pi}_j]}{\partial x_j}\delta\tau$$

In the result of these transformations (8) will take the following form

$$[\vec{r}, \rho\delta\tau \frac{d\vec{v}}{dt}] + \frac{d}{dt}[\vec{r}, \sum_i m_i\vec{v}_i'] + \frac{d}{dt}\sum_i [\vec{r}_i', m_i\vec{v}] + \frac{d}{dt}\sum_i [\vec{r}_i', m_i\vec{v}_i'] =$$

$$=[\vec{r},\vec{F}\rho\delta\tau]+\sum_i[\vec{r}_i',m_i\vec{F}_i]+\sum_{j=1}^{3}\frac{\partial[\vec{r},\vec{\pi}_j]}{\partial x_j}\delta\tau+\oiint_{\sigma_{\delta\tau}}\sum_k[\vec{r}_{\sigma k}',\vec{\pi}_{nk}\sigma_k]$$

Be dividing this expression by $\delta\tau$ and having done regrouping, we'll obtain

$$[\vec{r},\rho\frac{d\vec{v}}{dt}]-[\vec{r},\rho\vec{F}]-\sum_{j=1}^{3}\frac{\partial}{\partial x_j}[\vec{r},\vec{\pi}_j]=$$

$$=-\frac{1}{\delta\tau}\{\frac{d}{dt}[\vec{r},\sum_i m_i\vec{v}_i']+\frac{d}{dt}\sum_i[\vec{r}_i',m_i\vec{v}]+\frac{d}{dt}\sum_i[\vec{r}_i',m_i\vec{v}_i']\}+$$

$$+\frac{1}{\delta\tau}(\sum_i[\vec{r}_i',m_i\vec{F}_i]+\oiint_{\sigma_{\delta\tau}}\sum_k[\vec{r}_{\sigma k}',\vec{\pi}_{nk}\sigma_k]) \qquad (10)$$

Right part of (10) is not equal to zero

$$-\frac{1}{\delta\tau}\{\frac{d}{dt}[\vec{r},\sum_i m_i\vec{v}_i']+\frac{d}{dt}\sum_i[\vec{r}_i',m_i\vec{v}]+\frac{d}{dt}\sum_i[\vec{r}_i',m_i\vec{v}_i']\}+$$

$$+\frac{1}{\delta\tau}(\sum_i[\vec{r}_i',m_i\vec{F}_i]+\oiint_{\sigma_{\delta\tau}}\sum_k[\vec{r}_{\sigma k}',\vec{\pi}_{nk}\sigma_k])\neq 0,$$

quod erat demonstrandum.

Accordingly, equation (6), which is used in textbooks of *Loitsyansky* /1/ and *Sedov* /2/ as theorem on change of moment of momentum **is fallacious**. Obviously from (10), that in formula (6) *Loitsyansky* /1/ and *Sedov* /2/ there can be no equality sign. In fact, due to (10) there is **inequality:**

$$[\vec{r},\rho\frac{d\vec{v}}{dt}]\neq[\vec{r},\rho\vec{F}]+\sum_{j=1}^{3}\frac{\partial}{\partial x_j}[\vec{r},\vec{\pi}_j]$$

It is widely known that theoretical conclusion about symmetry of strain tensor is established on the basis of equality (6). **Since (6) does not apply due to (10), there is no symmetry of strains tensor in continuous medium in the general case, quod erat demonstrandum.**

Note. In equilibrium, velocities of all particles of fluid are equal to

zero $\vec{v}_i = 0 \quad \forall i$, $\vec{v} = 0$, $\dfrac{d\vec{v}}{dt} = 0$, continuum equilibrium equation

$$\rho\vec{F} + \sum_{j=1}^{3} \frac{\partial \vec{\pi}_j}{\partial x_j} = 0\,, \tag{12}$$

is equality to zero of major force (see **§8)**, besides, the expression (11) will be equal to zero

$$\frac{1}{\delta\tau}\{\frac{d}{dt}[\vec{r}, \sum_i m_i \vec{v}_i'] + \frac{d}{dt}\sum_i [\vec{r}_i', m_i \vec{v}] + \frac{d}{dt}\sum_i [\vec{r}_i', m_i \vec{v}_i']\} = 0 \tag{13}$$

By fundamental determination of moment of momentum (7) will be equal to zero in the right part (10) and expression

$$\frac{1}{\delta\tau}(\sum_i [\vec{r}_i', m_i \vec{F}_i] + \oiint_{\sigma_{\delta\tau}} \sum_k [\vec{r}_{\sigma k}', \vec{\pi}_{nk}\sigma_k]) = 0 \tag{14}$$

due to equalities (13) and (14) in equilibrium, there will be equiality of major moment of foreces to zero

$$\vec{M} = [\vec{r}, \rho\vec{F}] + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}[\vec{r}, \vec{\pi}_j] = 0 \tag{15}$$

From (12) results equality to zero of moment of major force:

$$\vec{M}_c = [\vec{r}, \rho\vec{F} + \sum_{j=1}^{3} \frac{\partial \vec{\pi}_j}{\partial x_j}] = 0 \tag{16}$$

By theorem 1 **§8** in the general case they are not equal to each other by definition: $\vec{M}_c \neq \vec{M}$. Their equality $\vec{M}_c = \vec{M}$ takes place under conditions of theorem 4:

1. *Strains are parallel to coordinate axes:*

$$\vec{\pi}_{ij} \parallel \frac{\partial \vec{r}}{\partial x_j}, \ \ \mathrm{j} = 1,2,3,$$

2. *Strains components are symmetrical:*

$$\pi_{ij} = \pi_{ji}, i, j = 1,2,3. \tag{17}$$

This implies logical derivation that *symmetry of strain tensor* of continuum even in equilibrium *is not a physical property of medium,* but is the result of *artificial* equating of the major

moment $\vec{M}$ to moment of major force $\vec{M}_c$, i.e. ***mathematical condition of*** **equality of these moments in the sense that if in some point of flow conditions (17) are observed simultaneously, the moment of major force becomes equal to major moment $\vec{M}_c = \vec{M}$, if these conditions (17) are not observed, they are not equal to each other: $\vec{M}_c \neq \vec{M}$.**

§10. Paradoxal application of theorems on angular momentum (momentum) variation. *Fallacy of* the *Navier-Stokes* equiations

Let's remember the following **logical order** of deriving **moment of momentum change theorem**. It is very important here. In particular, that from law of conservation of momentum of individual material points (see /7/)

$$\frac{d(m_i\vec{v}_i)}{dt} = \vec{F}_i + \sum_{k=1 k\neq i}^{N} \vec{F}_{ik}, \quad i = 1,\dots,N \qquad (1)$$

results, firstly, law of conservation of material points system momentum (see /7/)

$$\sum_{i=1}^{N} \frac{d(m_i\vec{v}_i)}{dt} = \sum_{i=1}^{N} \vec{F}_i + \sum_{i=1}^{N} \sum_{k=1 k\neq i}^{N} \vec{F}_{ik}, \quad i = 1,\dots,N, \qquad (2)$$

where sum of internal forces is equal to zero

$$\sum_{i=1}^{N} \sum_{k=1 k\neq i}^{N} \vec{F}_{ik} = 0,$$

therefore, there is theorem on change of momentum for the system of material points

$$\frac{d}{dt} \sum_{i=1}^{N} m_i\vec{v}_i = \sum_{i=1}^{N} \vec{F}_i, \quad i = 1,\dots,N, \qquad (3)$$

secondly, (1) results in moment of momentum for each individual material point

$$\frac{d}{dt}[\vec{r}_i, m_i\vec{v}_i] = [\vec{r}_i, \vec{F}_i] + \sum_{k=1 k\neq i}^{N} [\vec{r}_i, \vec{F}_{ik}], \quad i = 1,\dots,N, \qquad (4)$$

summing up of which results for the system of material points in equality

$$\frac{d}{dt}\sum_{i=1}^{N}[\vec{r}_i, m_i\vec{v}_i] = \sum_{i=1}^{N}[\vec{r}_i, \vec{F}_i] + \sum_{i=1}^{N}\sum_{k=1 k\neq i}^{N}[\vec{r}_i, \vec{F}_{ik}], \quad i = 1,\ldots,N ,$$

where sum of moments of internal forces is equal to zero

$$\sum_{i=1}^{N}\sum_{k=1 k\neq i}^{N}[\vec{r}_i, \vec{F}_{ik}] = 0$$

Due to this theorem on change of moment of momentum in closed system of material points above is formulated as follows

$$\frac{d}{dt}\sum_{i=1}^{N}[\vec{r}_i, m_i\vec{v}_i] = \sum_{i=1}^{N}[\vec{r}_i, \vec{F}_i], \tag{5}$$

where $\sum_{i=1}^{N}[\vec{r}_i, \vec{F}_i]$ - sum of moments of external forces.

Basing on (5), precise formulation of theorem on change of moment of momentum in lementary volume $\delta\tau$ of continuum was given above:

$$\frac{d}{dt}\sum_{i=1}^{N}[\vec{r}_i, m_i\vec{v}_i] = \sum_{i=1}^{N}[\vec{r}_i, m_i\vec{F}_i] + \oint\!\!\!\oint_{\sigma_{\delta\tau}}\sum_{k}[\vec{r}_{\sigma k}, \vec{\pi}_{nk}\sigma_k] \tag{6}$$

So, (1) results in (4) and then (5), *but not vice versa, i.e. (5) does not result in (1) explicitly*. Let's apply this sequence to deriving theorem on change of moment of momentum in continuous medium in other formulations. Let's represent continuum dynamics equation under strains

$$\rho\frac{d\vec{v}}{dt} = \rho\vec{F} + \frac{\partial\vec{\pi}_x}{\partial x} + \frac{\partial\vec{\pi}_y}{\partial y} + \frac{\partial\vec{\pi}_z}{\partial z} \tag{7}$$

As was shown in inductive method **§4, (**this equation is the prototype of equation (3)). Multiplying (7) vectorially by radius-vector $\vec{r}$, we'll derive a prototype of moments theorem (5):

$$[\vec{r}, \rho\frac{d\vec{v}}{dt}] = [\vec{r}, \rho\vec{F}] + [\vec{r}, \frac{\partial\vec{\pi}_x}{\partial x} + \frac{\partial\vec{\pi}_y}{\partial y} + \frac{\partial\vec{\pi}_z}{\partial z}] \tag{8}$$

From equivalent transformation (8)

$$\frac{d}{dt}[\vec{r},\rho\vec{v}]-[\vec{r},\vec{v}\,\frac{d\rho}{dt}]=[\vec{r},\rho\vec{F}]+[\vec{r},\frac{\partial\vec{\pi}_x}{\partial x}+\frac{\partial\vec{\pi}_y}{\partial y}+\frac{\partial\vec{\pi}_z}{\partial z}] \quad (9)$$

results theorem on change of moment of momentum $\rho\vec{v}$, due to a unit of volume:

$$\frac{d}{dt}[\vec{r},\rho\vec{v}]=[\vec{r},\vec{v}\,\frac{d\rho}{dt}]+[\vec{r},\rho\vec{F}]+[\vec{r},\frac{\partial\vec{\pi}_x}{\partial x}+\frac{\partial\vec{\pi}_y}{\partial y}+\frac{\partial\vec{\pi}_z}{\partial z}] \quad (10)$$

For incompressible continuous media $\rho = const$ and theorem on change of moment of momentum has the following form

$$\frac{d}{dt}[\vec{r},\rho\vec{v}]=[\vec{r},\rho\vec{F}]+[\vec{r},\frac{\partial\vec{\pi}_x}{\partial x}+\frac{\partial\vec{\pi}_y}{\partial y}+\frac{\partial\vec{\pi}_z}{\partial z}], \quad (11)$$

fully coinsiding with moment definition (5). Moment of individual volume $\delta\tau$ is equal to $\rho\delta\tau\vec{v}=\delta m\vec{v}$, therefore moment of momentum change theorem from (8) is derived for elementary volume $\delta\tau$ in the following form

$$[\vec{r},\rho\delta\tau\,\frac{d\vec{v}}{dt}]=[\vec{r},\rho\delta\tau\vec{F}]+[\vec{r},\frac{\partial\vec{\pi}_x}{\partial x}+\frac{\partial\vec{\pi}_y}{\partial y}+\frac{\partial\vec{\pi}_z}{\partial z}]\delta\tau$$

or

$$[\vec{r},\delta m\,\frac{d\vec{v}}{dt}]=[\vec{r},\delta m\vec{F}]+[\vec{r},\frac{\partial\vec{\pi}_x}{\partial x}+\frac{\partial\vec{\pi}_y}{\partial y}+\frac{\partial\vec{\pi}_z}{\partial z}]\delta\tau$$

For compressible media $\delta\tau=\delta\tau(t)$ and $\frac{d\delta\tau}{dt}=\delta\tau div\vec{v}$, $\frac{d\delta m}{dt}=0$ and this expression transforms into theorem on change of moment of momentum for individual volume $\delta\tau$ for compressible media

$$\frac{d}{dt}[\vec{r},\delta m\vec{v}]=[\vec{r},\delta m\vec{F}]+[\vec{r},\frac{\partial\vec{\pi}_x}{\partial x}+\frac{\partial\vec{\pi}_y}{\partial y}+\frac{\partial\vec{\pi}_z}{\partial z}]\delta\tau \quad (12)$$

Integration (12) for the total volume τ results in expression

$$\iiint_\tau \frac{d}{dt}[\vec{r}, \rho\delta\tau\vec{v}] = \iiint_\tau [\vec{r}, \rho\delta\tau\vec{F}] +$$

$$+ \iiint_\tau [\vec{r}, \frac{\partial\vec{\pi}_x}{\partial x} + \frac{\partial\vec{\pi}_y}{\partial y} + \frac{\partial\vec{\pi}_z}{\partial z}]\delta\tau \,, \qquad (13)$$

in principle different from familiar formula

$$\iiint_\tau \frac{d}{dt}[\vec{r}, \vec{v}\rho\delta\tau] = \iiint_\tau [\vec{r}, \vec{F}\rho\delta\tau] + \oiint_\sigma [\vec{r}, \vec{\pi}_n\delta\sigma], \qquad (14)$$

given in books of *Loitsyansky*, *Sedov* and other**. Obviously, conclusion of symmetry of strain tensor in no way comes from correctly formulated theorem on moment of momentum change (12).** Fallacy of (14) and formulae derived from it was discussed in previous paragraphs.

Well, list above paradoxes and evidences **dissymmetry of strain tensor** of continuum confirm **mistaken of *Stokes* hypothesis**. As exactly by this hypothesis derived **equations of *Navier-Stokes,* really, equation of *Stokes***

$$\rho\frac{dv_i}{dt} + \frac{\partial p}{\partial x_i} = \rho F_i + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}[\mu(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})] - \frac{2}{3}\frac{\partial}{\partial x_i}(\mu div\vec{v}), i = 1,2,3,$$

$$\rho c_v\frac{dT}{dt} = div(\lambda gradT) + \frac{\mu}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j})^2 -$$

$$- p div\vec{v} - \frac{2}{3}\mu(div\vec{v})^2$$

are mistaken.

Equations of ductile incompressible fluid, set ***Navier*** by dissymmetry of strain tensor of *Newton* $\pi_н$,

$$\rho[\frac{\partial\vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v}] + \nabla p = \mu\Delta\vec{v} + \rho\vec{F},$$

$$(\nabla, \vec{v}) = 0$$

are rectilineal.

§11. Analogue of *Stokes* hypothesis. Antisymmetric tensor strains

§7 set absolutely logical issue: **if, according to *Stokes* hypothesis, symmetric strains**

$$\pi_{ji(c)} = -(p + 2/3\mu div\vec{v})\delta_{ji} + \mu(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}), i, j = 1,2,3 \qquad (1)$$

are caused by deformative displacements $\sum_{j=1}^{3}\frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})\delta x_j$**,**

what strains create in *Taylor* series (in convective acceleration)

$$\delta v_i = \sum_{j=1}^{3}[\frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) + \frac{1}{2}(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})]\delta x_j, i = 1,2,3 \qquad (2)$$

rotational displacements $\frac{1}{2}[rot\vec{v}, \delta\vec{r}] = \sum_{j=1}^{3}\frac{1}{2}(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})\delta x_j$?

An *alternative hypothesis* is put forward to answer this question by analogy with *Stokes* hypothsis*:* let *tangential stress be proportional to components of rotational displacement*, i.e. there are *antisymmetric* (skew-symmetric) strains $\pi^*_{ji} = \frac{1}{2}\mu(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}),$

accordingly, alternative tensor will be equal to

$$\pi_{(Alt)} = -pE + 2\mu\tilde{S},$$

$$\pi_{ji(Alt)} = -p\delta_{ij} + \mu(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}), i, j = 1,2,3 \qquad (3)$$

Substituting (3) into equiations of dynamics in strains, we obtain *alternative equations* of viscous flud dynamics:

$$\rho\frac{dv_i}{dt} + \frac{\partial p}{\partial x_i} = \rho F_i + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}(\mu(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})), \ i = 1,2,3 \qquad (4)$$

Normal strains $\pi_{ii(Alt)} = -p, i = 1,2,3$ correspond to *Pascal* law. The following fact are of interest. With constant viscosity $\mu = const$ equations (4) are represented as follows

$$\rho\frac{dv_i}{dt}+\frac{\partial p}{\partial x_i}=\rho F_i+\mu\sum_{j=1}^{3}\frac{\partial^2 v_i}{\partial x_j^2}-\mu\frac{\partial}{\partial x_i}(\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_j}),i=1,2,3, \quad (5)$$

from which for incompressible fluid without sources and due to continuity equation $div\vec{v}\equiv\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_j}=0$ obtained are ***Navier* equations** for incompressible fluid

$$\rho\frac{dv_i}{dt}+\frac{\partial p}{\partial x_i}=\rho F_i+\mu\sum_{j=1}^{3}\frac{\partial^2 v_i}{\partial x_j^2},\ \ \mathrm{i}=1,2,3 \quad (6)$$

The same ***Navier*** equations are also derived from *Stokes* equations *with symmetric strains* (1)*:*

$$\rho\frac{dv_i}{dt}+\frac{\partial p}{\partial x_i}=\rho F_i+\sum_{j=1}^{3}\frac{\partial}{\partial x_j}[\mu(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i})]-\frac{2}{3}\frac{\partial}{\partial x_i}(\mu div\vec{v}),i=1,2,3, \quad (7)$$

and the same equations (6) are obtained from equations of dynamics *with nosymmetric Newton strains***:**

$$\pi_{н}=-[p+(\mu/3-\mu')div\vec{v})]E+\mu\bar{S}\ , \quad (8)$$

$$\rho\frac{dv_i}{dt}+\frac{\partial p}{\partial x_i}=\rho F_i+\sum_{j=1}^{3}\frac{\partial}{\partial x_j}(\mu\frac{\partial v_i}{\partial x_j})-\frac{\partial}{\partial x_i}[(\mu/3-\mu')div\vec{v}],i=1,2,3$$

But *Stokes* hypothesis (1) leads to the above-given paradoxes, because, as was proved, continuum strans tensor in the general case *is nonsymmetric*, and in *alternative hypothesis* equations (4) in the general cases of variable viscosity $\mu\neq const$ loose property of ellipticity. These two circumstances confirm adequacy of nonsymmetric *Newton* strain tensor (8) to flow of viscous fluids.

§12. New equations of vicous fluids dynamics with Newton nonsymmetric strain tensor

$$\pi_{ji(н)}=-[p+(\frac{1}{3}\mu-\mu')div\vec{v}]\delta_{ji}+\mu\frac{\partial v_i}{\partial x_j},i,j=1,2,3$$

Substituting components of *Newton* tensor given in **§**7 in Cartesian coordinates, into the general equation in strains and projecting to axes of coordinates we derive new equations of viscous fluid dynamics (*Jakupov K.B.* /8/):

$$\rho(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z})+\frac{\partial p}{\partial x}=$$

$$=\frac{\partial}{\partial x}(\mu\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y})+\frac{\partial}{\partial z}(\mu\frac{\partial u}{\partial z})-\frac{\partial}{\partial x}[(\frac{1}{3}\mu-\mu')div\vec{v}]+\rho F_x,$$

$$\rho(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z})+\frac{\partial p}{\partial y}=$$

$$=\frac{\partial}{\partial x}(\mu\frac{\partial v}{\partial x})+\frac{\partial}{\partial y}(\mu\frac{\partial v}{\partial y})+\frac{\partial}{\partial z}(\mu\frac{\partial v}{\partial z})-\frac{\partial}{\partial y}[(\frac{1}{3}\mu-\mu')div\vec{v}]+\rho F_y,$$

$$\rho(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z})+\frac{\partial p}{\partial z}=$$

$$=\frac{\partial}{\partial x}(\mu\frac{\partial w}{\partial x})+\frac{\partial}{\partial y}(\mu\frac{\partial w}{\partial y})+\frac{\partial}{\partial z}(\mu\frac{\partial w}{\partial z})-\frac{\partial}{\partial z}[(\frac{1}{3}\mu-\mu')div\vec{v}]+\rho F_z$$

from the total equation of energy balances thermal conductivity equation is derived

$$\rho c_v(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z})=$$

$$=\frac{\partial}{\partial x}(\lambda\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(\lambda\frac{\partial T}{\partial y})+\frac{\partial}{\partial z}(\lambda\frac{\partial T}{\partial z})-p\,div\vec{v}-(\frac{1}{3}\mu-\mu')(div\vec{v})^2+$$

$$+\mu[(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2+(\frac{\partial u}{\partial z})^2+(\frac{\partial v}{\partial x})^2+$$

$$+(\frac{\partial v}{\partial y})^2+(\frac{\partial v}{\partial z})^2+(\frac{\partial w}{\partial x})^2+(\frac{\partial w}{\partial y})^2+(\frac{\partial w}{\partial z})^2] \tag{2}$$

In incompressible fluid $\rho = const$, $div\vec{v}=0$:

$$\rho(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z})+\frac{\partial p}{\partial x}=$$

$$=\frac{\partial}{\partial x}(\mu\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y})+\frac{\partial}{\partial z}(\mu\frac{\partial u}{\partial z})+\rho F_x,$$

$$\rho(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z})+\frac{\partial p}{\partial y}=$$

$$= \frac{\partial}{\partial x}(\mu \frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(\mu \frac{\partial v}{\partial y}) + \frac{\partial}{\partial z}(\mu \frac{\partial v}{\partial z}) + \rho F_y, \qquad (3)$$

$$\rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) + \frac{\partial p}{\partial z} =$$

$$= \frac{\partial}{\partial x}(\mu \frac{\partial w}{\partial x}) + \frac{\partial}{\partial y}(\mu \frac{\partial w}{\partial y}) + \frac{\partial}{\partial z}(\mu \frac{\partial w}{\partial z}) + \rho F_z,$$

$$\rho c_V(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}) = \frac{\partial}{\partial x}(\lambda \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(\lambda \frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(\lambda \frac{\partial T}{\partial z}) +$$

$$+ \mu[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2 + (\frac{\partial v}{\partial x})^2 +$$

$$+ (\frac{\partial v}{\partial y})^2 + (\frac{\partial v}{\partial z})^2 + (\frac{\partial w}{\partial x})^2 + (\frac{\partial w}{\partial y})^2 + (\frac{\partial w}{\partial z})^2] \qquad (4)$$

Derivation of ***Navier*** equations for incompressible fluids

$$\rho \frac{dv_i}{dt} + \frac{\partial p}{\partial x_i} = \rho F_i + \mu \sum_{j=1}^{3} \frac{\partial^2 v_i}{\partial x_j^2}, i = 1,2,3 \qquad (5)$$

from *Stokes* equations in constant viscosity

$$\rho \frac{dv_i}{dt} + \frac{\partial p}{\partial x_i} = \rho F_i + \mu \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}), i = 1,2,3,$$

as is known, uses continuity equation $div\vec{v} = 0$ /1/. If the flow contains discretely located sources or drains with specific power J, $div\vec{v} = J$ will be a discontinuous nondifferentiable function, therefore derivation of the indicated equations in the form (5) is impossible, while (5) from new equations (3) us derived without the use of continuity equation $div\vec{v} = 0$.

As opposed to *Navier-Stokes* equations, equations (1) contain 9 derivatives less in three-dimensional case and 4 less in two-dimensional. Besides, equations (1) are easily represented in divergent form:

$$\frac{\partial \rho v_i}{\partial t} + div(\rho v_i \vec{v}) + \frac{\partial p}{\partial x_i} =$$

$$= \rho F_i + div(\mu grad v_i) - \frac{\partial}{\partial x_i}[(\frac{1}{3}\mu - \mu')div\vec{v}], i = 1,2,3 \quad (6)$$

If in the right part of this equation $div\vec{v} = 0$ is included, derived is analogous divergent expression of equations (3) of incompressible fluid.

Brief expression of energy balances equation for nonsymmetric tensor

$$\rho c_v \frac{dT}{dt} = div(\lambda gradT) +$$

$$+ \mu \sum_{i=1}^{3} \sum_{j=1}^{3} (\frac{\partial v_i}{\partial x_j})^2 - p div\vec{v} - (\frac{1}{3}\mu - \mu')(div\vec{v})^2 \quad (7)$$

in divergent representation is similar with dynamics equation (6)

$$c_v[\frac{\partial \rho T}{\partial t} + div(\rho T\vec{v})] = div(\lambda gradT) +$$

$$+ \mu \sum_{i=1}^{3} \sum_{j=1}^{3} (\frac{\partial v_i}{\partial x_j})^2 - p div\vec{v} - (\frac{1}{3}\mu - \mu')(div\vec{v})^2$$

In new equations of dynamics, attention is attracted by dissipative members $div(\mu grad v_i)$, *i=1,2,3*, **which in their structure** are similar with dissipative member in energy balance equation $div(\lambda gradT)$, derived from basic *Fourier* conduction law. *Prandtl* number $\Pr = \frac{c_p \mu}{\lambda}$ connects viscosity μ with heat conduction of medium λ, which emphasizes uniform essence of molecular transport of substances, which in one case is the temperature *T,* and in other cases – components of velocity $v_i, i = 1,2,3$ or concentration C_m according to *Fick* law.

Note. Due to arbitrariness of choise of second coefficient of viscosity, if $\mu' = \frac{1}{3}\mu$ is included, the given equations will be reduced by 12 derivatives.

§13. New equations of vicous fluids dynamics with Newton nonsymmetric strain tensor in cylindrical coordinates

$$\pi_{и} = -[p + (1/3\mu - \mu')div\vec{v}]E + \mu\bar{S}$$

Deriving similar equations in cylindrical and spherical coordinate systems does not result in any difficulties, the only thing here is to take into account nonsymmtry of strains in these systems (*Jakupov K.B.* /8/).

In cylindrical system for constant viscosity $\mu = const$ and density, dynamics equations coinside with known *Navier-Stokes* equations /3/, in the case of variable viscosity and density, they are derived by substituting with given in **§**7 nonsymmetric strains

$$\pi_{r\varepsilon} = \mu(\frac{\partial v_\varepsilon}{\partial r} - \frac{v_\varepsilon}{r}),\ \ \pi_{\varepsilon r} = \frac{\mu}{r}\frac{\partial v_r}{\partial \varepsilon},\ \ \pi_{zr} = \mu\frac{\partial v_r}{\partial z},$$

$$\pi_{rz} = \mu\frac{\partial v_z}{\partial r}, \pi_{z\varepsilon} = \mu\frac{\partial v_\varepsilon}{\partial z},\ \ \pi_{\varepsilon z} = \frac{\mu}{r}\frac{\partial v_z}{\partial \varepsilon}.$$

$$\pi_{rr} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v}] + \mu\frac{\partial v_r}{\partial r},$$

$$\pi_{\varepsilon\varepsilon} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v}] + \mu[\frac{1}{r}\frac{\partial v_\varepsilon}{\partial \varepsilon} + \frac{v_r}{r}],$$

$$\pi_{zz} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v}] + \mu\frac{\partial v_z}{\partial z},$$

in continuum dynamics equation (*Lykov A.V.* /3/):

$$\rho(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\varepsilon}{r}\frac{\partial v_r}{\partial \varepsilon} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\varepsilon^2}{r}) = \frac{1}{r}\frac{\partial(r\pi_{rr})}{\partial r} +$$

$$+\frac{1}{r}\frac{\partial \pi_{\varepsilon r}}{\partial \varepsilon} + \frac{\partial \pi_{zr}}{\partial z} - \frac{\pi_{\varepsilon\varepsilon}}{r} + \rho F_r,$$

$$\rho(\frac{\partial v_\varepsilon}{\partial t} + v_r\frac{\partial v_\varepsilon}{\partial r} + \frac{v_\varepsilon}{r}\frac{\partial v_\varepsilon}{\partial \varepsilon} + v_z\frac{\partial v_\varepsilon}{\partial z} + \frac{v_r v_\varepsilon}{r}) =$$

$$= \frac{1}{r^2}\frac{\partial (r^2 \pi_{r\varepsilon})}{\partial r} + \frac{1}{r}\frac{\partial \pi_{\varepsilon\varepsilon}}{\partial \varepsilon} + \frac{\partial \pi_{z\varepsilon}}{\partial z} + \rho F_\varepsilon ,$$

$$\rho(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\varepsilon}{r}\frac{\partial v_z}{\partial \varepsilon} + v_z \frac{\partial v_z}{\partial z}) =$$

$$= \frac{1}{r}\frac{\partial (r\pi_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \pi_{\varepsilon z}}{\partial \varepsilon} + \frac{\partial \pi_{zz}}{\partial z} + \rho F_z$$

energy balance equation takes the following form

$$\rho c_v(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\varepsilon}{r}\frac{\partial T}{\partial \varepsilon} + v_z \frac{\partial T}{\partial z}) = \frac{1}{r}\frac{\partial}{\partial r}(\lambda r \frac{\partial T}{\partial r}) +$$

$$+ \frac{1}{r^2}\frac{\partial}{\partial \varepsilon}(\lambda \frac{\partial T}{\partial \varepsilon}) + \frac{\partial}{\partial z}(\lambda \frac{\partial T}{\partial z}) + \Phi_v ,$$

$$\Phi_v = \mu[(\frac{\partial v_r}{\partial r})^2 + (\frac{1}{r}\frac{\partial v_\varepsilon}{\partial \varepsilon} + \frac{v_r}{r})^2 + (\frac{\partial v_z}{\partial z})^2 + (\frac{\partial v_r}{\partial z})^2 +$$

$$+ (\frac{\partial v_z}{\partial r})^2 + (\frac{1}{r}\frac{\partial v_r}{\partial \varepsilon})^2 + (\frac{\partial v_\varepsilon}{\partial r} - \frac{v_\varepsilon}{r})^2 +$$

$$+ (\frac{\partial v_\varepsilon}{\partial z})^2 + (\frac{1}{r}\frac{\partial v_z}{\partial \varepsilon})^2] - p\,div\vec{v} - (\frac{1}{3}\mu - \mu')(div\vec{v})^2$$

Note. Due to arbitrariness of selection of the second coefficient of viscosity, if $\mu' = \mu / 3$ is included, the given equations will be reduced not less than by 12 derivatives.

§14. New equations of vicous fluids dynamics with Newton’s nonsymmetric strain tensor in spherical coordinates

$$\pi_{н} = -[p + (1/3\mu - \mu')div\vec{v}]E + \mu\bar{S}$$

Components of nonsymmetric strain tensor according to Newton law have the following form (*Jakupov K.B.*/8 /):

$$\pi_{\phi\varepsilon} = \mu \frac{1}{r\sin\varepsilon}\frac{\partial v_\varepsilon}{\partial \phi}, \pi_{\varepsilon\phi} = \mu \frac{\sin\varepsilon}{r}\frac{\partial}{\partial \varepsilon}(\frac{v_\phi}{\sin\varepsilon}), \pi_{r\varepsilon} = \mu(\frac{\partial v_\varepsilon}{\partial r} - \frac{v_\varepsilon}{r}),$$

$$\pi_{\varepsilon r} = \frac{\mu}{r}\frac{\partial v_r}{\partial \varepsilon}, \pi_{rr} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v}] + \mu \frac{\partial v_r}{\partial r},$$

$$\pi_{\varepsilon\varepsilon} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v}] + \mu(\frac{1}{r}\frac{\partial v_\varepsilon}{\partial \varepsilon} + \frac{v_r}{r}),$$

$$\pi_{\phi r} = \mu\frac{1}{r\sin\varepsilon}\frac{\partial v_r}{\partial \phi}, \pi_{r\phi} = \mu r\frac{\partial}{\partial r}(\frac{v_\phi}{r}),$$

$$\pi_{\phi\phi} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v}] +$$

$$+\mu(\frac{1}{r\sin\varepsilon}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\varepsilon ctg\varepsilon}{r}),$$

$$div\vec{v} \equiv \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\varepsilon}\frac{\partial v_\phi}{\partial \phi} + \frac{1}{r\sin\varepsilon}\frac{\partial(v_\varepsilon \sin\varepsilon)}{\partial \varepsilon},$$

are substituted into equations of dynamics in strains (*Lykov A.V.* /3/):

$$\rho(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\varepsilon}{r}\frac{\partial v_r}{\partial \varepsilon} + \frac{v_\phi}{r\sin\varepsilon}\frac{\partial v_r}{\partial \phi} - \frac{v_\varepsilon^2 + v_\phi^2}{r}) =$$

$$= \frac{1}{r^2}\frac{\partial(r^2\pi_{rr})}{\partial r} + \frac{1}{r\sin\varepsilon}\frac{\partial(\pi_{\varepsilon r}\sin\varepsilon)}{\partial \varepsilon} +$$

$$+\frac{1}{r\sin\varepsilon}\frac{\partial \pi_{\phi r}}{\partial \phi} - \frac{\pi_{\varepsilon\varepsilon} + \pi_{\phi\phi}}{r} + \rho F_r,$$

$$\rho(\frac{\partial v_\varepsilon}{\partial t} + v_r\frac{\partial v_\varepsilon}{\partial r} + \frac{v_\varepsilon}{r}\frac{\partial v_\varepsilon}{\partial \varepsilon} + \frac{v_\phi}{r\sin\varepsilon}\frac{\partial v_\varepsilon}{\partial \phi} + \frac{v_r v_\varepsilon}{r} - \frac{v_\phi^2 ctg\varepsilon}{r}) =$$

$$= \frac{1}{r^2}\frac{\partial(r^2\pi_{r\varepsilon})}{\partial r} + \frac{1}{r\sin\varepsilon}\frac{\partial(\pi_{\varepsilon\varepsilon}\sin\varepsilon)}{\partial \varepsilon} +$$

$$+\frac{1}{r\sin\varepsilon}\frac{\partial \pi_{\phi\varepsilon}}{\partial \phi} + \frac{\pi_{r\varepsilon} + \pi_{\varepsilon r}}{r} - \frac{ctg\varepsilon}{r}\pi_{\phi\phi} + \rho F_\varepsilon,$$

$$\rho(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\varepsilon}{r}\frac{\partial v_\phi}{\partial \varepsilon} + \frac{v_\phi}{r\sin\varepsilon}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\varepsilon v_\phi}{r}ctg\varepsilon) =$$

$$= \frac{1}{r^2}\frac{\partial(r^2\pi_{r\phi})}{\partial r} + \frac{1}{r}\frac{\partial \pi_{\varepsilon\phi}}{\partial \varepsilon} + \frac{1}{r\sin\varepsilon}\frac{\partial \pi_{\phi\phi}}{\partial \phi} +$$

$$+ \frac{\pi_{r\phi} + \pi_{\phi r}}{r} + \frac{2ctg\varepsilon}{r}(\pi_{\varepsilon\phi} + \pi_{\phi\varepsilon}) + \rho F_r,$$

in energy balance equation the following is included for nonsymmetric tensor

$$\Phi_v = \mu\{(\frac{\partial v_r}{\partial r})^2 + (\frac{1}{r}\frac{\partial v_\varepsilon}{\partial \varepsilon} + \frac{v_r}{r})^2 + (\frac{1}{r\sin\varepsilon}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\varepsilon ctg\varepsilon}{r})^2 +$$

$$+[r\frac{\partial}{\partial r}(\frac{v_\varepsilon}{r})]^2 + (\frac{1}{r}\frac{\partial v_r}{\partial \varepsilon})^2 + [\frac{\sin\varepsilon}{r}\frac{\partial}{\partial \varepsilon}(\frac{v_\phi}{\sin\varepsilon})]^2 + (\frac{1}{r\sin\varepsilon}\frac{\partial v_\varepsilon}{\partial \phi})^2 +$$

$$+(\frac{1}{r\sin\varepsilon}\frac{\partial v_r}{\partial \phi})^2 + [r\frac{\partial}{\partial r}(\frac{v_\phi}{r})]^2\} - p\,div\vec{v} - (\frac{1}{3}\mu - \mu')(div\vec{v})^2$$

Note. Due to arbitrariness of selection of the second coefficient of viscosity, if $\mu' = \frac{1}{3}\mu$ is included, the given equations will be reduced not less than by 12 derivatives.

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Chapter 2. FALLACY OF PRANDTL EQUATIONS IN BOUNDARY-LAYER THEORY

Fallacy of theory proposed by *Prandtl* in 1904, states that in system of equations of boundary-layer derived from reductions /1/, /2/ carried by him, **second *Newton* law and law of conservation of mass are not observed.**

§1. Velocity vector components are determined with the help of Newton's second law or momentum theorem

This is the truth that became trivial, but here appeared necessity in its application for proving statements given in the chapter title. Let's use the following formula of Second *Newton* law for a material point with mass $m=const$, that is under resultant force $\vec{f}$

$$m\frac{d\vec{v}}{dt} = \vec{f}$$

For force $\vec{f}$, not depending on velocity $\vec{v} = u\vec{i} + v\vec{j} + w\vec{k}$, computation of all velocity components is performed with the help of integration of second law on time slice $[t_0, t]$

$$\vec{v}\big|_t = \vec{v}\,|_{t_0} + \int_{t_0}^{t} \vec{f} dt / m,$$

which gives in projections

$$u\big|_t = u\,|_{t_0} + \int_{t_0}^{t} f_x dt / m, \quad v\big|_t = v\,|_{t_0} + \int_{t_0}^{t} f_y dt / m, \quad w\big|_t = w\,|_{t_0} + \int_{t_0}^{t} f_x dt / m,$$

In case of force $\vec{f}$, depending on $\vec{v}$, integration becomes more complicated, which in general takes place while computation of flows of viscous fluid.

Let's consider *Navier* equation of viscous incompressible fluid

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \mu \Delta \vec{v} + \rho \vec{F},$$

having multiplied it by elementary volume $\delta\tau$, we'll come to formulation of *Newton* second law for mass $\delta m = \rho \delta \tau$:

$$\delta m \frac{d\vec{v}}{dt} = \vec{f},$$

where $\vec{f} = (-\nabla p + \mu \Delta \vec{v} + \rho \vec{F})\delta m$ is the major (resultant) force.

It is obvious that according to Newton second law, velocity vector $\vec{v}$***, and, consequently, all its components must be computed only from the major equation of dynamics*** $\delta m \frac{d\vec{v}}{dt} = \vec{f}$***, and not from equation of law of conservation of mass*** $div\vec{v} = 0$, ***as is the case in Prandtl equations.***

The purpose, for which ***law of conservation of mass*** is used in the form of continuity equation, is considerd below.

§2. Action of law of mass conservation upon flow is performed via pressure only

Let's consider equation of viscous incompressible fluid dynamics

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v} + \frac{1}{\rho}\nabla p = \nu\Delta\vec{v} + \vec{F}, \qquad (1)$$

along with continuity equation

$$(\nabla, \vec{v}) = 0 \qquad (2)$$

In **§1** it was showed that computation of velocity components must be carried out only through solving of *Navier* equations-(1) in projections, accordingly, logically, ***pressure p must be computed only from equation of law of conservation of mass, in other words, from continuity equation (2)***, which, properly speaking, is realized in grid solution methods of initially-boundary value problems for the system (1), (2) (see /3/).

Technique of deriving equation determining pressure is known for long and nalitically is in the following. Differentiating continuity equation in time *t*, nedded expression is derived

$$(\nabla, \frac{\partial \vec{v}}{\partial t}) = 0 \qquad (3)$$

In the result of substituting into (3) a time derivative

$$\frac{\partial \vec{v}}{\partial t} = \nu\Delta\vec{v} + \vec{F} - (\vec{v}, \nabla)\vec{v} - \frac{1}{\rho}\nabla p$$

Found is desired analytical expression for pressure, ***realizing fundamental connection between pressure and velocity***

$$\Delta p + (\nabla, (\vec{v}, \nabla)\vec{v}) - (\nabla, \rho\vec{F}) - (\nabla, \mu\Delta\vec{v}) = 0 \qquad (4)$$

It is obtained ***in natural manner*** from *law of conservation of mass.*

Equation (4) is apparently an elliptical equation in relation to pressure, which thus emphasizes *diffusive* character of pressure distribution in continuous medium. In Cartesian system, due to equation (2) it is significantly reduced becase equality to zero of the last member in the left part (4) $(\nabla, \mu\Delta\vec{v}) = \mu\Delta(\nabla, \vec{v}) = 0$ accordingly, there is a more compact equation

$$\Delta p + (\nabla, (\vec{v}, \nabla)\vec{v}) - (\nabla, \rho\vec{F}) = 0, \qquad (5)$$

where there are no forces of medium viscosity, i.e. this equation is equally just for both – viscous and ideal fluids.

Similar association is present in compressible media as well. From divergent dynamics equations /3/

$$\frac{\partial\rho v_i}{\partial t}+div(\rho v_i\vec{v})+\frac{\partial p}{\partial x_i}= \qquad (6)$$

$$=\rho F_i+div(\mu grad v_i)-\frac{\partial}{\partial x_i}[(\frac{1}{3}\mu-\mu')div\vec{v}],i=1,2,3,$$

$$c_v[\frac{\partial\rho T}{\partial t}+div(\rho T\vec{v})]=div(\lambda grad T)+$$

$$+\mu\sum_{i=1}^{3}\sum_{j=1}^{3}(\frac{\partial v_i}{\partial x_j})^2-pdiv\vec{v}-(\frac{1}{3}\mu-\mu')(div\vec{v})^2,$$

$$\frac{\partial\rho}{\partial t}+\sum_i\frac{\partial\rho v_i}{\partial x_i}=0,\ p=\rho RT$$

are singled out from (6) time derivatives

$$\frac{\partial\rho v_i}{\partial t}=-div(\rho v_i\vec{v})-\frac{\partial p}{\partial x_i}+\rho F_i+$$

$$+div(\mu grad v_i)-\frac{\partial}{\partial x_i}[(\frac{1}{3}\mu-\mu')div\vec{v}],i=1,2,3 \qquad (7)$$

and are substituted into differentiated by t equations of law of conservation of mass

$$\frac{\partial^2\rho}{\partial t^2}+\sum_i\frac{\partial^2\rho v_i}{\partial x_i\partial t}=0, \qquad (8)$$

and because of which hyperbolic-type equation is derived for pressure

$$\frac{\partial^2}{\partial t^2}(\frac{p}{RT})=\sum_i\Big\{\frac{\partial^2 p}{\partial x_i^2}+\frac{\partial^2}{\partial x_i^2}[(\frac{1}{3}\mu-\mu')div\vec{v}]+$$

$$+\frac{\partial}{\partial x_i}[div(\rho v_i\vec{v})-div(\mu grad v_i)-\rho F_i]\Big\}, \qquad (9)$$

i.e. in gases, pressure distribution has a wave-like behavior. In numerical methods of solution of initially-boundary value problems for systems (1),(2) or (6) computed pressures, as a rule, are carried out by difference analogues of equation (4) or equation (9), derived

from difference analogues of continuity equations (See /3/). At this, immediate approximation of equation (4) by *homogenous* difference schemes with the use in the capacity of boundary conditions on boundary σ of dynamics equations (1):

$$\left.\frac{\partial p}{\partial n}\right|_{\sigma} = (\nabla p, \vec{n})\Big|_{\sigma} = (\{-\rho[\frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v}] + \mu\Delta\vec{v} + \rho\vec{F}\}\Big|_{\sigma}, \vec{n}),$$

as was shown by numerical experiments, is not an effective method for solution of initially-boundary value problems for system (1), (2) (iteration algorithms for computation of pressure will be vary). Absolutely similar situation takes place also for gas dynamics equations (6). Therefore, in numerical solution of systems of equations (1), (2) or (6) used is a bit different computational technique of fundamental associations relatiozations (4) or (9). At this substantial is that depending on net domain nodes location, difference analogues of associations (4) or (9) are derived in the form of closed system of *inhomogeneous* difference equations for pressure, which are easily realized by suitable iteration algorithm, for instance, simple but convergent parametric iteration method /3/.

§3. *Fallacy* of *Prandtl* equations is in incomplete observance of second *Newton* law and due to absence of action of law of conservation of mass upon flow via pressure

Exact equations, describing viscous flows, including those in boundary layer are undoubtfully equations (1),(2) for incompressible fluids and equations (6) for gases.

Ludwig von Prandtl supposed that in boundary layer near rigid surface there are estimates $y \approx \sqrt{\nu}$, $v \approx \sqrt{\nu}$, $u \approx 1, x \approx 1$, basing on which he discarded in equations (1) for plane flows of viscous fluid

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}), \qquad (10)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}), \qquad (11)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad (12)$$

relatively small members and introduced simplified system /2/

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2}, \qquad (13)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y}=0, \qquad (14)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (15)$$

The main error of Prandtl system of equations (13), (14), (15) according to **§**1, is in neglect of projection of second *Newton* law to *y-axis*, which is confirmed by equality to zero in (14), i.e. second *Newton* law is not observed in full, preserved only projection (13) of the second *Newton* law to axis x.

Obviously, for equations (13), (14), (15) analogue of *fundamental* association (4) will be equation for pressure in the following form

$$\frac{\partial^2 p}{\partial x^2}-\frac{\partial^2 v}{\partial y\partial t}+\rho\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})-\mu\frac{\partial^3 u}{\partial x\partial y^2}=0 \qquad (16)$$

According to above-stated concept of solution of initially-boundary value problems for equations (1), (2) or (6), during realization considered is system (13), (14), (16). (14) results in dependence of pressure on x and t only: $p=p(x,t)$. Bearing this in mind, system (13), (14), (16) may be represented in the form of two equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2}, \qquad (17)$$

$$\frac{\partial^2 p}{\partial x^2}-\frac{\partial^2 v}{\partial y\partial t}+\rho\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})-\mu\frac{\partial^3 u}{\partial x\partial y^2}=0 \qquad (18)$$

Thus, for 3 unknown functions u,v,p derived are two equations (17), (18) only. Obviously, one more equation for computation of lateral velocity v is missing.

Continuity equation (15) may not be used for defining v, since it is

derived from equation (18), at this, at the initial time *t=0* there must be an equality

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\Bigg|_{t=0}=0.$$

Accoring to §1 transverse component of velocity *v* must be computed from projection of the second *Newton* law to axis *y* (11). *But in this equation, Prandtl neglected all members associated with* velocity component *v*

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\ \nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0,$$

and left equality to zero (14), which was said above.

Vital error of Prandtl is that according to his **concept in the form of equations (13),(14) major law of dynamics, in this case second *Newton* law, act in the flow in boundary layer only in longitudinal direction along axis x, in crosswise direction along axis *y do not act,*** accordingly, ***in the direction of y –axis there must be no motion according to major law of dynamics***, and if there is no motion in crosswise direction, this means that there is no apprpopriate velocity copponent, i.e. $v=0$.

If so, *Prandtl* equations (13),(14),(15) are reduced to form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2},\ \frac{1}{\rho}\frac{\partial p}{\partial y}=0,\ \frac{\partial u}{\partial x}=0$$

and final form of *Prandtl* equations will be as follows

$$\frac{\partial u}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2},\frac{1}{\rho}\frac{\partial p}{\partial y}=0$$

The last equation indicates that pressure will be a function onle *x* and *t:* $p=p(x,t)$*, and also* $u=u(x,t)$.

In the result, for two unknown functions only one equation. Obtained contradiction, is apparently *connected with the fact* that *Prandtl system neglects projection of second Newton law to* ***y axis.***

Equating to zero of acceleration $\frac{\partial \vec{v}}{\partial t}+u\frac{\partial \vec{v}}{\partial x}+v\frac{\partial \vec{v}}{\partial y}+w\frac{\partial \vec{v}}{\partial z}=0$ in steady flows is justified by the following considerations: that spatial

members of transfer $u\frac{\partial\vec{v}}{\partial x}+v\frac{\partial\vec{v}}{\partial y}+u\frac{\partial\vec{v}}{\partial z}$ are negligibly small. Of course, there can be stationarity, but products in $u\frac{\partial\vec{v}}{\partial x}+v\frac{\partial\vec{v}}{\partial y}+w\frac{\partial\vec{v}}{\partial z}$ may be pretty large. Let's show it on *analogue* of only one of products of this expression of transfer, for example, on $v\frac{dv}{dy}$. Let $v=\varepsilon\sin\alpha y$, where $\alpha = const >> 1$ is a very large but finite number, $\varepsilon -$ parameter providing for infinitesimality of function v, its derivative is equal to $\frac{\partial v}{\partial y}=\alpha\varepsilon\cos\alpha y$. ***Therefore product*** $v\frac{\partial v}{\partial y}=\alpha\varepsilon^2\sin\alpha y\cos\alpha y$ will be an infinite nonzero number, it is enough to take $\alpha = const * \varepsilon^{-2}$. The same will happen for the whole class of functions with removable discontinuities. ***Thus, equating to zero of acceleration***

$$\frac{\partial\vec{v}}{\partial t}+u\frac{\partial\vec{v}}{\partial x}+v\frac{\partial\vec{v}}{\partial y}+w\frac{\partial\vec{v}}{\partial z}=0$$

is not always justified**. Unjustified first of all because of contradiction with the major law of dynamics with second Newton law, which states: effect of force causes acceleration which is nonzero.**

Logical demand of strict observance of the second *Newton* law gives instead of fallacious *Prandtl* system (13),(14),(15) the following approximate system:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}=\nu\frac{\partial^2 v}{\partial y^2},$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

This system is also obtained from *Navier* equations, based on carried estimations, according to which all members of the second equation of this system of one infinitesimal order:

$$\frac{\partial v}{\partial t}\approx\sqrt{\nu},\quad u\frac{\partial v}{\partial x}\approx\sqrt{\nu},\quad v\frac{\partial v}{\partial y}\approx\sqrt{\nu},\ \ \nu\frac{\partial^2 v}{\partial y^2}\approx\sqrt{\nu}\ ,$$

and neglected transverse pressure gradient of a higher infinitesimal order $\frac{1}{\rho}\frac{\partial p}{\partial y}\approx\nu$, therefore it may be neglected if desired.

In short, neglected was only this pressure gradient, and all the rest members of equation (11) are preserved as values of equal infinitesimal order, thus, second *Newton* law is observed partially, i.e. in both directions, as opposed to fallacious system of *Prandtl* (13),(14),(15).

Obvisouly for these equations, analogue of *fundamental* association (4) will be equation for pressure in the following form

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\rho\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})-\mu\frac{\partial^3 u}{\partial x\partial y^2}+$$

$$+\rho\frac{\partial}{\partial y}(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y})-\mu\frac{\partial^3 v}{\partial y^3}=0$$

This equation may be reduced to the following form

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\rho\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})+\rho\frac{\partial}{\partial y}(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y})=0\,,$$

having in view continuity equation.

In our system for three unknown functions *u,v,p* there are corresponding three equations, at this second *Newton* law and law of conservation of mass are observed, and also satisfied is fundamental association (4), as opposed to *Prandtl* equations (13) (14) (15).

Steady-state *Prandtl* equations

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2}\,, \tag{19}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y}=0\,, \tag{20}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{21}$$

have the same deficiencies that are in (13) (14) (15).

Due to (20) $p = p(x)$ and dynamics equation (19) is represented as follows

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{dp}{dx}=\nu\frac{\partial^2 u}{\partial y^2} \tag{22}$$

By analogy with (13) (14) (15) here also for 3 unknown functions *u,v,p* exist only 2 equations (21) and (22) equation for computation of lateral *v* is missing, since according to concept (4) in a steady-state case from (22) we can single out

$$u=(-v\frac{\partial u}{\partial y}-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2})/\frac{\partial u}{\partial x} \tag{23}$$

and, substituting into equation of law of conservation of mass (21), hence obtain equation for pressure in the following form

$$\frac{\partial}{\partial x}[(-v\frac{\partial u}{\partial y}-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2})/\frac{\partial u}{\partial x}]+\frac{\partial v}{\partial y}=0 \tag{24}$$

Equation (24) forms system along with dynamics equation (22) and again for 3 unknown functions *u,v,p* there are only 2 equations (22) and (24), at this continuity equation (21) is the result of equations (22) and (24).

As is known /1/,/2/, Prandtl, for computation of pressure p, does not use this equation (24) ***or*** (18)***, thus broken is the fundamental association between pressure and velocity.*** Bearing in mind (20) or (14), i.e. dependency of pressure only on *x* and *t*: *p=p(x,t)*, *Prandtl* considers equation (17) in external incoming flow, considering it as moving with velocity

$$u = U_\infty(x,t)\,,\ v=0,$$

and obtains for pressure gradient relation

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial U_\infty}{\partial t} - U_\infty \frac{\partial U_\infty}{\partial x} \tag{25}$$

(This *Prandtl* technique, resulted in (25), resembles action of baron *Münchhausen*, who pulled himself and his horse from bog pulling up hair on his head).

Obviously approach (25) of *Prandtl* contradicts to equation of fundamental association (18). In steady-state flows, assuming homogeneity of external flow

$$u = U_\infty = \text{const},\ v = 0,$$

Prandtl, from (25) derives equality to zero of pressure gradient $\frac{dp}{dx} = 0$, consequently, according to *Prandtl*, in boundary layer pressure turns out to be constant throughout the flow $p=const$ and equations (13) (14) (15) are reduced to limit

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{26}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{27}$$

In this system (26),(27) longitudinal velocity *u* is determined from second *Newton* law (26), transverse velocity *v* is determined from continuity equation (27), which is fully in contradiction to given in §1, §2, §3 proofs.

Fallacy of *Prandtl* equations (26), (27) was confirmed by numerical solution of *Navier* equations (10), (11), (12) by semi-implicit scheme /3/, used for computation of longitudinal flow for a plate with length *L* in the area $[0 \le x \le L, 0 \le y \le \delta]$, transferred into dimensionless area $[0 \le x' \le 1, 0 \le y' \le 8/\sqrt{\text{Re}}]$. Results of computations, carried out with number $\text{Re} = U_\infty L/\nu = 10^5$ on grid $N_x = 20, N_y = 20$, showed that in boundary layer of the disc pressure, there is function of both variables *x,y*, pressure gradients differ from zero $\frac{\partial p}{\partial x} \ne 0, \frac{\partial p}{\partial y} \ne 0$. Let's give table of pressure values

in three cross sections of the disc 1) $x'=0.05$, 2) $x'=1-0.1$, 3) $x'=1-0.05$, $0 < y' \leq 8/\sqrt{\mathrm{Re}}$:

1) 0.982 0.982 0.982 0.982 0.981 0.980 0.979 0.978 0.978 0.979 0.980 0.981 0.983 0.985 0.988 0.990 0.994 0.997 0.998 1.000

2) 0.923 0.923 0.924 0.924 0.925 0.926 0.927 0.928 0.929 0.930 0.932 0.933 0.935 0.937 0.938 0.940 0.942 0.944 0.946 0.947

3) 0.903 0.903 0.903 0.903 0.903 0.903 0.903 0.903 0.903 0.903 0.903 0.903 0.904 0.904 0.904 0.904 0.904 0.904 0.904 0.904

Distribution of longitudinal component of velocity *u* in cross section $x'=1$, $0 \leq y' \leq 8/\sqrt{\mathrm{Re}}$

0.000 0.090 0.176 0.259 0.339 0.417 0.493 0.566 0.636 0.700 0.760 0.811 0.855 0.892 0.922 0.946 0.964 0.977 0.986 0.991 1.000

differs from values obtained from solution of *Prandtl* equations with $U_\infty = const$

0.000 0.108 0.216 0.324 0.429 0.530 0.625 0.711 0.784 0.845 0.893 0.929 0.954 0.971 0.983 0.990 0.994 0.996 0.998 0.999 1.000

In flows of incompressible fluid due to deceleration of particles on solid streamlined surface on external edge of boundary layer the flow will not be homogenious $u \neq U_\infty, v \neq 0$, transverse component of velocity will differ from zero but extremum condition will be just

$$\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0, \left.\frac{\partial v}{\partial y}\right|_{y=\delta} = 0\,.$$

So, approach of *Prandtl* to computation of pressure by formula (25), and steady-state system itself (26),(27) contradict with the equation of fundamental association (16) between pressure and velocity, and this means that law of conservation of mass is not observed.

§4. Laminar boundary layers in compressible flow

All the above-mentioned paradoxes are present in equations of laminar boundary layer in compressible flow /1/:

$$\rho(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})+\frac{\partial p}{\partial x}=\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y}),$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y}=0,\ \frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0,$$

$$\rho c_p(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y})=\frac{\partial}{\partial y}(\lambda\frac{\partial T}{\partial y})+u\frac{\partial p}{\partial x}+\mu(\frac{\partial u}{\partial y})^2$$

Correct, i.e. not violating the second *Newton* law and law of conservation of mass, will be the following equations

$$\rho(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})+\frac{\partial p}{\partial x}=\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y}),$$

$$\rho(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y})+\frac{\partial p}{\partial y}=\frac{\partial}{\partial y}(\mu\frac{\partial v}{\partial y}),\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0,$$

$$\rho c_p(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y})=\frac{\partial}{\partial y}(\lambda\frac{\partial T}{\partial y})+u\frac{\partial p}{\partial x}+\mu(\frac{\partial u}{\partial y})^2$$

§5. Three-dimensional boundary layers

Obviously equations of three-dimensional boundary layer /1/

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2},$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial z}=\nu\frac{\partial^2 w}{\partial y^2},$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y}=0,\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

have the same disadvantages that are peculiar for (13), (14), (15). These equations must be replaced with the following ones

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2},$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial y}=\nu\frac{\partial^2 v}{\partial y^2}, \qquad (28)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial z}=\nu\frac{\partial^2 w}{\partial y^2},\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

but the most objective will be use of full equations of viscous fluid dynamics /2/.

§6. Substitution of *Blasius* equation

Thus, pressure gradients are not equal to zero and in apprpoximation of *Prandtl* even in steady-state flow, equations must have the form (19)

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{29}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y}=0, \tag{30}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{31}$$

Differentiated (29) by *y*

$$u\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+v\frac{\partial^2 u}{\partial y^2}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial^2 p}{\partial y\partial x}=\nu\frac{\partial^3 u}{\partial y^3} \tag{32}$$

In equation (32) due to *Prandtl* assumption (30)

$$\frac{\partial^2 p}{\partial y\partial x}=\frac{\partial^2 p}{\partial x\partial y}=0, \tag{33}$$

therefore, from (32) derived is third order equation

$$u\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+v\frac{\partial^2 u}{\partial y^2}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}=\nu\frac{\partial^3 u}{\partial y^3} \tag{34}$$

Introduction of flow function

$$u=\partial\psi/\partial y, v=-\partial\psi/\partial x$$

allows precise integration of continuity equation (31) and from (34) results equation of forth order for ψ :

$$\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial y^2\partial x}+\frac{\partial^2\psi}{\partial y^2}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y^3}-\frac{\partial^2\psi}{\partial y\partial x}\frac{\partial^2\psi}{\partial y^2}=\nu\frac{\partial^4\psi}{\partial y^4} \tag{35}$$

Let’s use self-simulated *Blasius* variables

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}, \psi = f(\eta)\sqrt{\nu x U_\infty}, u = U_\infty f'(\eta), v = \frac{1}{2}(\eta f' - f)\sqrt{\frac{\nu U_\infty}{x}}$$

As the result we arrive to ordinary differential equation of 4^{th} order

$$2f'''' + ff''' + f'f'' = 0 \qquad (36)$$

As opposed to third-order *Blasius* equation $2f''' + ff'' = 0$. Weakness of *Blasius* equation is absence of pressure gradient: $dp/dx = 0$. In principle, from (36) results equality $2f''' + ff'' = \frac{1}{\rho}\frac{dp}{dx}$, but here $dp/dx \neq 0$! For equation (36) needed are four boundary conditions, which corresponds to equation (35), while for *Blasius* equation there were three such conditions, which means loss of an important physical condition of flow.

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Chapter 3. ABOUT TURBULENT FLOWS SIMULATION

Problem of cloing *Raynolds* equations is widely known. Various semiempirical theories, major of which are given in monograph /1/ are used for practical purposes. These models are very remotely related to original equations for averaged functions and contain a number of functional inclusions and constants with indefinite physical properties and doubtful universality. Many semiempirical theories, like *mixing length theory* (Mischungweg) of *Prandtl* /2/, simulate averaged turbulent flows in boundary layer by equations which lack pressure gradient, thus law of conservation of mass and

second *Newton* law are not observed, which was discussed in detail in chapter 2.

Of course, appropriate results from such theories may be obtained only if slanting coefficients.

§1. Paradox of *Reynolds* averaging formula

In Raynolds approach to derivation of equations for averaged in time values, actual values of hydrothermodynamic functions f are represented in the form of averaged $\bar{f}$ and fluctuating components f':

$$f = \bar{f} + f', \bar{f} = \frac{1}{t^o} \int_{t-\frac{t^o}{2}}^{t+\frac{t^o}{2}} f(x,y,z,\tau)d\tau, \bar{f}' = \frac{1}{t^o} \int_{t-\frac{t^o}{2}}^{t+\frac{t^o}{2}} f'(x,y,z,\tau)d\tau = 0 \quad (1)$$

While averaging nonlinear equations of *Navier-Stokes*, energy balance, diffusion etc. appear additional, so called *Reynolds* or fluctuating strains, heat and diffusion flows of the following form

$$\overline{\Lambda\{f'\psi'\}} = \frac{1}{t^o} \int_{t-\frac{t^o}{2}}^{t+\frac{t^o}{2}} \Lambda\{f'(x,y,z,\tau)\psi'(x,y,z,\tau)\}d\tau \quad (2)$$

($\Lambda -$differentiation operator), definition of which creates artificial problem of closing *Raynolds* equations.

Paradoxicality averaging formulae (1), (2) is in limits of integration $[t - t^o/2, t + t^o/2]$, where t^o - averaging time, and t - current time. According to *Raynolds* (1) it comes that preliminary it is necessary to know values of functions for future moments in time $[t, t + t^o/2]$ and only then carry out averagings by (1), which is not real. In computing experiments there is always possibility to carry out averaging to current moment of time t, using collected in memory values of function in previous moments of time:

$$f = \bar{f} + f', \bar{f} = \frac{1}{t^o} \int_{t-t^o}^{t} f(x,y,z,\tau)d\tau, \bar{f}' = \frac{1}{t^o} \int_{t-t^o}^{t} f'(x,y,z,\tau)d\tau = 0 \quad (3)$$

According to averaging (3) there is

$$\overline{\Lambda\{f'\psi'\}} = \frac{1}{t^o}\int_{t-t^o}^{t}\Lambda\{f'(x,y,z,\tau)\psi'(x,y,z,\tau)\}d\tau \qquad (4)$$

§2. Approximate calculation of averaging integrals

2.1. There are several simple formulae for approximate calculation of definite integrals. For example, in averaging by *Raynolds* (2) integral may be approximately calculated by average rectangular formula

$$\overline{\Lambda\{f'\psi'\}} = \frac{1}{t^o}\int_{t-\frac{t^o}{2}}^{t+\frac{t^o}{2}}\Lambda\{f'(x,y,z,\tau)\psi'(x,y,z,\tau)\}d\tau = \qquad (5)$$

$$= \Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)\} + \frac{(t^o)^2}{24}\frac{\partial^2}{\partial t^2}\Lambda\{f'(x,y,z,t^*)\psi'(x,y,z,t^*)\},$$

$$t^* \in [t - \frac{t^o}{2}, t + \frac{t^o}{2}]$$

By *Lagrange* formula

$$\frac{\partial^2}{\partial t^2}\Lambda\{f'(x,y,z,t^*)\psi'(x,y,z,t^*)\} = \frac{\partial^2}{\partial t^2}\Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)\} +$$

$$+ (t^* - t)\frac{\partial^3}{\partial t^3}\Lambda\{f'(x,y,z,t^{**})\psi'(x,y,z,t^{**})\} \qquad (6)$$

$$t^{**} \in [t, t^*] \quad or \quad t^{**} \in [t^*, t]$$

Substituting (6) into (5) we have

$$\overline{\Lambda\{f'\psi'\}} = \Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)\} + \qquad (7)$$

$$+ \frac{(t^o)^2}{24}\frac{\partial^2}{\partial t^2}\Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)\} + O(\frac{(t^o)^2}{24}|t^* - t|)$$

Taking into account that period of averaging t^o - one of the most important characteristics of turbulent flow and is determined in real flows as quantity comparatively small in relation to characteristic for this process time scale, in formula (7) residual member

$O(\frac{(t^o)^2}{24}|t^* - t|)$ may be neglected as relatively small quantity of third order, having in mind inequality $|t^* - t| \le t^o$.

In the result from (7) derived is approximation formula

$$\overline{\Lambda\{f'\psi'\}} = \Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)\} + \\ + \frac{(t^o)^2}{24}\frac{\partial^2}{\partial t^2}\Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)\}, \qquad (8)$$

having error

$$O(\frac{(t^o)^2}{24}|t^* - t|) = O(\frac{(t^o)^3}{24})$$

The same we do for averaged third and higher moments

$$\overline{\Lambda\{f'\psi'..\varphi'\}} = \Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)..\varphi'(x,y,z,t)\} + \\ + \frac{(t^o)^2}{24}\frac{\partial^2}{\partial t^2}\Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)..\varphi'(x,y,z,t)\} \qquad (9)$$

2.2. In real averaging (4) for approximate calculation of integral it is possible to apply rectangular formula

$$\overline{\Lambda\{f'\psi'\}} = \frac{1}{t^o}\int_{t-t^o}^{t}\Lambda\{f'(x,y,z,\tau)\psi'(x,y,z,\tau)\}d\tau = \\ = \Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)\} - \\ - \frac{t^o}{2}\frac{\partial}{\partial t}\Lambda\{f'(x,y,z,t^*)\psi'(x,y,z,t^*)\}, \qquad (10)$$

where $t^* \in [t - t^o, t]$.

By *Lagrange* formula

$$\frac{\partial}{\partial t}\Lambda\{f'(x,y,z,t^*)\psi'(x,y,z,t^*)\} = \frac{\partial}{\partial t}\Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)\} + \\ + (t^* - t)\frac{\partial^2}{\partial t^2}\Lambda\{f'(x,y,z,t^{**})\psi'(x,y,z,t^{**})\} \qquad (11)$$

$$t^{**} \in [t, t^*] \quad \text{или} \quad t^{**} \in [t^*, t]$$

Substituting (11) into (10) , we’ll find

$$\overline{\Lambda\{f'\psi'\}} = \frac{1}{t^o}\int_{t-t^o}^{t}\Lambda\{f'(x,y,z,\tau)\psi'(x,y,z,\tau)\}d\tau =$$

$$= \Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)\} -$$

$$-\frac{t^o}{2}\frac{\partial}{\partial t}\Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)\} + O(\frac{t^0}{2}\left|t^* - t\right|) \qquad (12)$$

From (12) approximate formula is derived

$$\overline{\Lambda\{f'\psi'\}} = \Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)\} -$$

$$-\frac{t^o}{2}\frac{\partial}{\partial t}\Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)\}, \qquad (13)$$

having error $O(\frac{t^o}{2}\,|\,t^* - t\,|) = O(\frac{t^{o\,2}}{2})$ due to inequality $|\,t^* - t\,| \le t^o$, since $t^* \in [t - t^o, t]$.

The same we do for averaged third and higher moments

$$\overline{\Lambda\{f'\psi'..\varphi'\}} = \Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)...\varphi'(x,y,z,t)\} -$$

$$-\frac{t^o}{2}\frac{\partial}{\partial t}\Lambda\{f'(x,y,z,t)\psi'(x,y,z,t)..\varphi'(x,y,z,t)\} \qquad (14)$$

Formulae (8), (9), (13), (14) allow closing systems of averaged in accordance with *Raynolds* equations of hydrodynamics with involving pulsations equations or equations for second moments $\overline{f'\psi'}$, $\overline{f'\varphi'}$, $\overline{\psi'\varphi'}$ etc.

§3. Application of pulsations equiation for closing *Reynolds* equations

In equations of viscous incompressible fluids

$$\rho[\frac{\partial v_i}{\partial t} + div(v_i\vec{v})] + \frac{\partial p}{\partial x_i} = \rho F_i + \mu\Delta v_i\,, i = 1,2,3,$$

$$\rho c_p[\frac{\partial T}{\partial t} + div(T\vec{v})] = \lambda\Delta T\ , \sum_i \frac{\partial v_i}{\partial x_i} = 0\,,$$

$$\frac{\partial C_m}{\partial t} + div(C_m \vec{v}) = D_m \Delta C_m, \quad m = 1,\ldots \tag{15}$$

$$v_i\Big|_S = \varphi_i, T\Big|_S = q, C_m\Big|_S = q_m, v_i\Big|_{t=0} = v_i^0, T\Big|_{t=0} = T^0, C_m\Big|_{t=0} = C_m^{\ 0}$$

let’s show use of formula (8).

Thanks to averaging by *Raynolds* (1), from (10) derived are equations for averaged functions

$$\rho\left(\frac{\partial \bar{v}_i}{\partial t} + \sum_j \rho \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j}\right) + \sum_j \frac{\partial \rho \overline{v'_j v'_i}}{\partial x_j} + \frac{\partial \bar{p}}{\partial x_i} = \rho \bar{F}_i + \mu \Delta \bar{v}_i, i = 1,2,3,$$

$$\rho c_p\left(\frac{\partial \bar{T}}{\partial t} + \sum_j \bar{v}_j \frac{\partial \bar{T}}{\partial x_j}\right) + c_p \sum_j \frac{\partial \overline{\rho v'_j T'}}{\partial x_j} = \lambda \Delta \bar{T}, \sum_i \frac{\partial \bar{v}_i}{\partial x_i} = 0,$$

$$\frac{\partial \bar{C}_m}{\partial t} + \sum_j \bar{v}_j \frac{\partial \bar{C}_m}{\partial x_j} + \sum_j \frac{\partial \overline{v'_j C'_m}}{\partial x_j} = D_m \Delta \bar{C}_m, m = 1,\ldots, \tag{16}$$

$$\bar{v}_i\Big|_S = \bar{\varphi}_i, \bar{T}\Big|_S = \bar{q}, \bar{C}_m\Big|_S = \bar{q}_m, \bar{v}_i\Big|_{t=0} = \bar{v}_i^0, \bar{T}\Big|_{t=0} = \bar{T}^0, \bar{C}_m\Big|_{t=0} = \bar{C}_m^{\ 0}$$

When subtracting equations (16) in appropriate manner from (15) obtained are pulsations equations

$$\rho \frac{\partial v'_i}{\partial t} + \sum_j \rho\left(\bar{v}_j \frac{\partial v'_i}{\partial x_j} + v'_j \frac{\partial v'_i}{\partial x_j} + v'_j \frac{\partial \bar{v}_i}{\partial x_j}\right) - \sum_j \frac{\partial \rho \overline{v'_j v'_i}}{\partial x_j} + \frac{\partial p'}{\partial x_i} =$$

$$= \rho F'_i + \mu \Delta v'_i, \ \boldsymbol{i} = 1{,}2{,}3, \sum_i \frac{\partial v'_i}{\partial x_i} = 0, \tag{17}$$

$$\rho c_p\left[\frac{\partial T'}{\partial t} + \sum_j \left(\bar{v}_j \frac{\partial T'}{\partial x_j} + v'_j \frac{\partial T'}{\partial x_j} + v'_j \frac{\partial \bar{T}}{\partial x_j}\right)\right] - c_p \sum_j \frac{\partial \overline{\rho v'_j T'}}{\partial x_j} = \lambda \Delta T',$$

$$\frac{\partial C'_m}{\partial t} + \sum_j \left(\bar{v}_j \frac{\partial C'_m}{\partial x_j} + v'_j \frac{\partial C'_m}{\partial x_j} + v'_j \frac{\partial \bar{C}_m}{\partial x_j}\right) - \sum_j \frac{\partial \overline{v'_j C'_m}}{\partial x_j} = D_m \Delta C'_m,$$

$$v'_i\Big|_S = \varphi'_i, T'\Big|_S = q', C'_m\Big|_S = q'_m, v'_i\Big|_{t=0} = v'^0_i, T'\Big|_{t=0} = T'^0, C'_m\Big|_{t=0} = C'_m{}^0$$

Further on, included into (16) and (17) pulsation expressions

$\sum_j \frac{\partial \rho \overline{v'_j v'_i}}{\partial x_j}$, $c_p \sum_j \frac{\partial \overline{\rho v'_j T'}}{\partial x_j}$, $\sum_j \frac{\partial \overline{v'_j C'_m}}{\partial x_j}$ are substituted by formula (13), in the result of which closed system of equations is obtained

$$\rho(\frac{\partial \bar{v}_i}{\partial t} + \sum_j \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} + \sum_j \frac{\partial v'_j v'_i}{\partial x_j} - \frac{t^o}{2} \sum_j \frac{\partial^2 v'_j v'_i}{\partial t \partial x_j}) + \frac{\partial \bar{p}}{\partial x_i} =$$

$$= \rho \bar{F}_i + \mu \Delta \bar{v}_i, \ \boldsymbol{i} = 1,2,3, \sum_i \frac{\partial \bar{v}_i}{\partial x_i} = 0, \qquad (18)$$

$$\rho c_p (\frac{\partial \bar{T}}{\partial t} + \sum_j \bar{v}_j \frac{\partial \bar{T}}{\partial x_j} + \sum_j \frac{\partial v'_j T'}{\partial x_j} - \frac{t^o}{2} \sum_j \frac{\partial^2 v'_j T'}{\partial t \partial x_j}) = \lambda \Delta \bar{T},$$

$$\frac{\partial \bar{C}_m}{\partial t} + \sum_j \bar{v}_j \frac{\partial \bar{C}_m}{\partial x_j} + \sum_j \frac{\partial v'_j C'_m}{\partial x_j} - \frac{t^o}{2} \sum_j \frac{\partial^2 v'_j C'_m}{\partial t \partial x_j} = D_m \Delta \bar{C}_m,$$

$$\rho[\frac{\partial v'_i}{\partial t} + \sum_j (\bar{v}_j \frac{\partial v'_i}{\partial x_j} + v'_j \frac{\partial \bar{v}_i}{\partial x_j}) + \frac{t^o}{2} \sum_j \frac{\partial^2 v'_j v'_i}{\partial t \partial x_j}] + \frac{\partial p'}{\partial x_i} =$$

$$= \rho F'_i + \mu \Delta v'_i, \quad \boldsymbol{i} = 1,2,3, \sum_i \frac{\partial v'_i}{\partial x_i} = 0, \qquad (19)$$

$$\rho c_p [\frac{\partial T'}{\partial t} + \sum_j (\bar{v}_j \frac{\partial T'}{\partial x_j} + v'_j \frac{\partial \bar{T}}{\partial x_j}) + \frac{t^o}{2} \sum_j \frac{\partial^2 v'_j T'}{\partial t \partial x_j}] = \lambda \Delta T',$$

$$\frac{\partial C'_m}{\partial t} + \sum_j (\bar{v}_j \frac{\partial C'_m}{\partial x_j} + v'_j \frac{\partial \bar{C}_m}{\partial x_j}) + \frac{t^o}{2} \sum_j \frac{\partial^2 v'_j C'_m}{\partial t \partial x_j} = D_m \Delta C'_m,$$

$$\bar{v}_i \Big|_S = \bar{\varphi}_i, \bar{T}\Big|_S = \bar{q}, \bar{C}_m \Big|_S = \bar{q}_m, \bar{v}_i \Big|_{t=0} = \bar{v}_i^0, \bar{T}\Big|_{t=0} = \bar{T}^0, \bar{C}_m \Big|_{t=0} = \bar{C}_m^0,$$

$$v'_i \Big|_S = \varphi'_i, T'\Big|_S = q', C'_m \Big|_S = q'_m, v'_i \Big|_{t=0} = v'^0_i, T'\Big|_{t=0} = T'^0, C'_m \Big|_{t=0} = C'^0_m$$

Sum of equations (18) and (19) is *Navier*, heat conduction and diffusion equations.

Averaging time t^o , organically included into closed system (18) and (19) along with such such physical properties of flow as the following coefficients – viscosity μ, heat conductivity λ, heat capacity c_p, diffusion D_m and density ρ, and also initial and boundary conditions for averaged and pulsations must be defined for each calculated flow. Numerical solution of closed intertwined system (18) and (19) for given μ, λ, c_p, ρ, D_m, $\overline{\varphi}_i, \varphi'_i, \overline{q}, q', \overline{q}_m, q'_m$ and initial time conditions does not represent difficulties in kind and may be carried out by difference schemes, developed and theoretically justified in /4/, /5/.

Equations for second moments $\overline{v'_i v'_j}$, $\overline{v'_j T'}$, $\overline{v'_j C'_m}$ may be composed with the use of *Keller-Friedman* method /1/. Let's limit ourselves with consideration of equations for second moments $\tau_{ij} = \overline{v'_i v'_j}$:

$$\rho[\frac{\partial \tau_{ij}}{\partial t} + \sum_k \overline{v}_k \frac{\partial \tau_{ij}}{\partial x_k} + \sum_k (\tau_{jk} \frac{\partial \overline{v}_i}{\partial x_k} + \tau_{ik} \frac{\partial \overline{v}_j}{\partial x_k})] + \overline{v'_j \frac{\partial p'}{\partial x_i}} + \overline{v'_i \frac{\partial p'}{\partial x_j}} =$$

$$= -\rho \sum_k \frac{\overline{\partial v'_i v'_j v'_k}}{\partial x_k} + \mu \Delta \tau_{ij} - 2\mu \sum_k \overline{\frac{\partial v'_i}{\partial x_k} \frac{\partial v'_j}{\partial x_k}} \qquad (20)$$

Expressing in system (20) second moments $\overline{v'_j \frac{\partial p'}{\partial x_i}}, \overline{v'_i \frac{\partial p'}{\partial x_j}}$, $\sum_k \overline{\frac{\partial v'_i}{\partial x_k} \frac{\partial v'_j}{\partial x_k}}$ by formula (13), and third moments $\sum_k \frac{\overline{\partial v'_i v'_j v'_k}}{\partial x_k}$ by formula (14) we'll obtain closed system along with equations (16) for $\overline{v}_i$ and equations (17) for v'_i, having set appropriate boundary $\tau_{ij}\big|_S$ and initial conditions $\tau_{ij}\big|_{t=0}$ for τ_{ij} :

$$\rho(\frac{\partial \overline{v}_i}{\partial t} + \sum_j \overline{v}_j \frac{\partial \overline{v}_i}{\partial x_j} + \sum_j \frac{\partial \tau_{ji}}{\partial x_j}) + \frac{\partial \overline{p}}{\partial x_i} = \rho \overline{F}_i + \mu \Delta \overline{v}_i, i = 1,2,3,$$

$$\sum_i \frac{\partial \bar{v}_i}{\partial x_i} = 0,$$

$$\rho[\frac{\partial v'_i}{\partial t} + \sum_j (\bar{v}_j \frac{\partial v'_i}{\partial x_j} + v'_j \frac{\partial v'_i}{\partial x_j} + v'_j \frac{\partial \bar{v}_i}{\partial x_j}) - \sum_j \frac{\partial \tau_{ij}}{\partial x_j}] + \frac{\partial p'}{\partial x_i} =$$

$$= \rho F'_i + \mu \Delta v'_i,\ i = 1,2,3,\ \sum_i \frac{\partial v'_i}{\partial x_i} = 0,$$

$$\rho[\frac{\partial \tau_{ij}}{\partial t} + \sum_k \bar{v}_k \frac{\partial \tau_{ij}}{\partial x_k} + \sum_k (\tau_{jk} \frac{\partial \bar{v}_i}{\partial x_k} + \tau_{ik} \frac{\partial \bar{v}_j}{\partial x_k})] + v'_j \frac{\partial p'}{\partial x_i} - \frac{t^o}{2} \frac{\partial}{\partial t} (v'_j \frac{\partial p'}{\partial x_i}) +$$

$$+ v'_i \frac{\partial p'}{\partial x_j} - \frac{t^o}{2} \frac{\partial}{\partial t} (v'_i \frac{\partial p'}{\partial x_j}) = -\rho \sum_k (\frac{\partial v'_i v'_j v'_k}{\partial x_k} - \frac{t^o}{2} \frac{\partial^2 v'_i v'_j v'_k}{\partial t \partial x_k}) + \mu \Delta \tau_{ij} -$$

$$- 2\mu \sum_k [\frac{\partial v'_i}{\partial x_k} \frac{\partial v'_j}{\partial x_k} - \frac{t^o}{2} \frac{\partial}{\partial t} (\frac{\partial v'_i}{\partial x_k} \frac{\partial v'_j}{\partial x_k})],$$

$$\bar{v}_i\Big|_S = \bar{\varphi}_i, \bar{v}_i\Big|_{t=0} = \bar{v}_i^0, v'_i\Big|_S = \varphi'_i, v'_i\Big|_{t=0} = v'^0_i, \tau_{ij}\Big|_S = \eta_{ij}, \tau_{ij}\Big|_{t=0} = \tau_{ij}^0,$$

In transition from (18) and (19) to equations in dimensionless variables, having taken as scales of – velocity V_*, linear dimensions *L*, time $t_* = L/V_*$, pressure $p_* = \rho V_*^2$, temperature T_*, obtained are known similarity criteria: numbers of: *Strouhal* - *Sh,* Euler - *Eu, Raynolds* - *Re, Froude* - *Fr, Prandtl* - *Pr, Schmidt* - *Sc*. Along with them one more dimensionless complex appears here - $Dg = \frac{t^0}{t_*}$ or

$Dg = \frac{t^0 V_*}{L}$, defining relation of averaging time to typical time scale.

§4. Turbulent flows simulation with account to friction caused by normal strains

As experiments show, turbulent flow has pulsating, irregular character with intense mixing of layers, which results in multiple increase of fluid flow resistance forces /1/. Basing on these facts, in

semiempirical theories, coefficient of turbulent viscosity μ_{T} is introduced, which by several orders exceeds coefficient of molecular viscosity μ, thus emphasizing role of increase of friction in developed turbulent flows. For friction force in laminar flows it is enough to determine tangential stress by *Newton* law with coefficient of molecular viscosity μ, and neglecting action of normal strains for friction (refer to **Chapter 1**). In turbulent flow friction increases. In this connection ***the following hypothesis is suggested*** (*Jakupov K.B.* /4/)**:** ***increase of friction forces in turbulent flows is connected with effect of action of normal strains upon tangential stress as a result of fluctuating mixing of layers that are moving irregularly, i.e. arises necessity to use known formula of friction of two contacting surfaces of continuums*** $F_{mp} = k_{mp} F_n$.

In laminar flows, in this formula, dimensionless friction coefficient "*k*" if equaled to zero: "*k=0"*.

This formula

$$F_{mp} = k_{mp} F_n,$$

where F_n - force acting normally to friction surfaces, in transition to tangential stress is transformed as follows.

Let strain $\vec{\pi}_i = \pi_{ii}\vec{i} + \pi_{ij}\vec{j} + \pi_{ik}\vec{k}$ act upon elemental area $\delta x_j \delta x_k$, lying on coordinate plane $ox_j x_k$, $j \neq k \neq i$,. According to above hypothesis, with the purpose of taking account of action of normal stress π_{ii} in tangential stress π_{ij}, π_{ik} along with friction by *Newton* law, the indicated strains are represented in the form of sum

$$\pi_{ij} = \pi_{ij(\text{н})} + \pi_{ij(T)}, \pi_{ik} = \pi_{ik(н)} + \pi_{ik(T)}, i \neq j, i \neq k, \qquad (21)$$

where $\pi_{ij(н)}$, $\pi_{ik(н)}$ are tangential stress of molecular transport defined by nonsymmetric formulae of *Newton* friction law in **Chapter 1**, and tangential stress

$\pi_{ij(T)}$, $\pi_{ik(T)}$ are contributions of action of normal stress and are calculated by friction formula $F_{mp} = k_{mp} F_n$ as follows. Let friction

force $\vec{F}_{mp}$ on area $\delta x_j \delta x_k$ has projections F_{mpij}, F_{mpik} for directions of orts $\vec{j}, \vec{k}$:

$$\vec{F}_{mp} = F_{mpij}\vec{j} + F_{mpik}\vec{k},$$

having divided by elemental area $\delta x_j \delta x_k$, we'll find tangential stress

$$\pi_{ij(T)} = \frac{F_{mpij}}{\delta x_j \delta x_k}, \pi_{ik(T)} = \frac{F_{mpik}}{\delta x_j \delta x_k} \quad i \neq j, i \neq k \qquad (22)$$

Meaning that in this case, force acting normally is equal to

$$F_n = \left|\pi_{ii}\delta x_j \delta x_k\right| = \left|\pi_{ii}\right|\delta x_j \delta x_k ,$$

friction formula is represented in the following form

$$\left|\vec{F}_{mp}\right| = kF_n = k\left|\pi_{ii}\right|\delta x_j \delta x_k$$

Component of velocity vector

$$\vec{v} = v_i\vec{i} + v_j\vec{j} + v_k\vec{k}$$

in area $\delta x_j \delta x_k$ is vector $\vec{v}_{jk} = v_j\vec{j} + v_k\vec{k}$.

Let vector $\vec{v}_{jk}$ make angles α_j and α_k accordingly with orts $\vec{j}, \vec{k}$, then cosines of these angles are obviously equal

$$\cos\alpha_j = v_j / \left|\vec{v}_{jk}\right|, \quad \cos\alpha_k = v_k \Big/ \left|\vec{v}_{jk}\right| \qquad (23)$$

$\left|\vec{v}_{jk}\right| = \sqrt{v_j^2 + v_k^2}$. But friction force $\vec{F}_{mp}$ is directed in opposite direction from velocity $\vec{v}_{jk}$: $\vec{v}_{jk} \uparrow\downarrow \vec{F}_{mp}$, therefore, there are formulae for this force components

$$F_{mpij} = \left|\vec{F}_{mp}\right|\cos\alpha_j = -k_{mp}\left|\pi_{ii}\right|\delta x_j \delta x_k \cos\alpha_j , \qquad (24)$$

$$F_{mpik} = \left|\vec{F}_{mp}\right|\cos\alpha_k = -k_{mp}\left|\pi_{ii}\right|\delta x_j \delta x_k \cos\alpha_k \qquad (25)$$

Substituting (24) and (25) into (22) and reducing $\delta x_j \delta x_k$, we'ss find

$$\pi_{ij(T)} = -k_{mp} \left| \pi_{ii} \right| \cos \alpha_j ,$$

$$\pi_{ik(T)} = -k_{mp} \left| \pi_{ii} \right| \cos \alpha_k \qquad (26)$$

Substituting cosines (23) into (26), represented via velocity components

$$\pi_{ij(T)} = -k_{mp} \left| \pi_{ii} \right| \cdot v_j (v_j^2 + v_k^2)^{-\frac{1}{2}}, \qquad (27)$$

$$\pi_{ik(T)} = -k_{mp} \left| \pi_{ii} \right| \cdot v_k (v_j^2 + v_k^2)^{-\frac{1}{2}}, i \neq j, i \neq k,$$

Thus, in formulae (21) tangential stress $\pi_{ij(T)}$, $\pi_{ik(T)}$ are defined in the form (27), and $\pi_{ij(н)}$, $\pi_{ik(н)}$ are determined by nonsymmetric formulae of Newton friction law in **Chapter 1**, which in Cartesian coordinates are equal to:

$$\pi_{11} = \pi_{xx}, \quad \pi_{22} = \pi_{yy}, \quad ,\pi_{33} = \pi_{zz},$$

$$\pi_{32(н)} = \mu \frac{\partial v}{\partial z}, \quad \pi_{12(н)} = \mu \frac{\partial v}{\partial x}, \quad \pi_{21(н)} = \mu \frac{\partial u}{\partial y},$$

$$\pi_{23(н)} = \mu \frac{\partial w}{\partial y}, \quad \pi_{31(н)} = \mu \frac{\partial u}{\partial z}, \quad \pi_{13(н)} = \mu \frac{\partial w}{\partial x}, \quad (28)$$

where normal strains are preserved in the initial form

$$\pi_{xx} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v}] + \mu \frac{\partial u}{\partial x},$$

$$\pi_{yy} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v}] + \mu \frac{\partial v}{\partial y}, \qquad (29)$$

$$\pi_{zz} = -[p + (\frac{1}{3}\mu - \mu')div\vec{v}] + \mu \frac{\partial w}{\partial z}$$

For pretty small values of molecular viscosity in (29) when *Raynolds* numbers are great, modules of normal strains may be assumed equal

$$|\pi_{ii}| = |-[p+(\frac{1}{3}\mu-\mu')div\vec{v}]+\mu\frac{\partial v_i}{\partial x_i}| = \qquad (30)$$

$$= p+(\frac{1}{3}\mu-\mu')div\vec{v}-\mu\frac{\partial v_i}{\partial x_i}, i=1,2,3$$

For simulating turbulent flows, tangential stress $\pi_{yx}, \pi_{xy}, \pi_{zx}, \pi_{xz}, \pi_{yz}, \pi_{zy}$ in equations of continuum dynamics

$$\rho(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}) = \rho F_x+\frac{\partial \pi_{xx}}{\partial x}+\frac{\partial \pi_{yx}}{\partial y}+\frac{\partial \pi_{zx}}{\partial z},$$

$$\rho(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}) = \rho F_y+\frac{\partial \pi_{xy}}{\partial x}+\frac{\partial \pi_{yy}}{\partial y}+\frac{\partial \pi_{zy}}{\partial z}, \qquad (31)$$

$$\rho(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}) = \rho F_z+\frac{\partial \pi_{xz}}{\partial x}+\frac{\partial \pi_{yz}}{\partial y}+\frac{\partial \pi_{zz}}{\partial z}$$

are determined by formulae (21) with account to expressions (27),(28),(29),(30)

$$\pi_{yx} = \mu\frac{\partial u}{\partial y}-k_{mp}|\pi_{yy}|\cdot u(u^2+w^2)^{-\frac{1}{2}},$$

$$\pi_{xy} = \mu\frac{\partial v}{\partial x}-k_{mp}|\pi_{xx}|\cdot v(v^2+w^2)^{-\frac{1}{2}},$$

$$\pi_{zx} = \mu\frac{\partial u}{\partial z}-k_{mp}|\pi_{zz}|\cdot u(u^2+v^2)^{-\frac{1}{2}},$$

$$\pi_{xz} = \mu\frac{\partial w}{\partial x}-k_{mp}|\pi_{xx}|\cdot w(v^2+w^2)^{-\frac{1}{2}},$$

$$\pi_{zy} = \mu \frac{\partial v}{\partial z} - k_{mp} |\pi_{zz}| \cdot v(u^2 + v^2)^{-\frac{1}{2}}, \quad (32)$$

$$\pi_{yz} = \mu \frac{\partial w}{\partial y} - k_{mp} |\pi_{yy}| \cdot w(u^2 + w^2)^{-\frac{1}{2}}$$

Substituting expressions (30) into (31), we obtain sought for equations for simulating both, laminar (*k=0*), as well as turbulent flows ($k \neq 0$):

$$\rho \frac{du}{dt} = \rho F_x + \frac{\partial}{\partial x}\{\mu \frac{\partial u}{\partial x} - [p + (\frac{1}{3}\mu - \mu')div\vec{v}]\} + \frac{\partial}{\partial y}\{\mu \frac{\partial u}{\partial y} -$$

$$- k[p + (\frac{1}{3}\mu - \mu')div\vec{v} - \mu \frac{\partial v}{\partial y}]u(u^2 + w^2)^{-\frac{1}{2}}\} + \frac{\partial}{\partial z}\{\mu \frac{\partial u}{\partial z} -$$

$$- k[p + (\frac{1}{3}\mu - \mu')div\vec{v} - \mu \frac{\partial w}{\partial z}]u(u^2 + v^2)^{-\frac{1}{2}}\},$$

$$\rho \frac{dv}{dt} = \rho F_y + \frac{\partial}{\partial x}\{\mu \frac{\partial v}{\partial x} - k[p + (\frac{1}{3}\mu - \mu')div\vec{v} - \mu \frac{\partial u}{\partial x}] \cdot$$

$$\cdot v(v^2 + w^2)^{-\frac{1}{2}}\} + \frac{\partial}{\partial y}\{\mu \frac{\partial v}{\partial y} - [p + (\frac{1}{3}\mu - \mu')div\vec{v}]\} +$$

$$+ \frac{\partial}{\partial z}\{\mu \frac{\partial v}{\partial z} - k[p + (\frac{1}{3}\mu - \mu')div\vec{v} - \mu \frac{\partial w}{\partial z}]v(u^2 + v^2)^{-\frac{1}{2}}\},$$

$$\rho \frac{dw}{dt} = \rho F_z + \frac{\partial}{\partial x}\{\mu \frac{\partial w}{\partial x} - k[p + (\frac{1}{3}\mu - \mu')div\vec{v} - \mu \frac{\partial u}{\partial x}] \cdot$$

$$\times w(v^2 + w^2)^{-\frac{1}{2}}\} + \frac{\partial}{\partial y}\{\mu \frac{\partial w}{\partial y} - k[p + (\frac{1}{3}\mu - \mu')div\vec{v} - \mu \frac{\partial v}{\partial y}] \cdot$$

$$\cdot w(u^2 + w^2)^{-\frac{1}{2}}\} + \frac{\partial}{\partial z}\{\mu \frac{\partial w}{\partial z} - [p + (\frac{1}{3}\mu - \mu')div\vec{v}]\};$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad (33)$$

continuity equation joins these equations as well

$$\frac{\partial \rho}{\partial t} + div(\rho \vec{v}) = 0 \qquad (34)$$

and energy balance equation (49) from **Chapter 1**

$$\rho \frac{d}{dt}(E + |\vec{v}|^2 / 2) = \qquad (35)$$

$$= \rho(\vec{F}, \vec{v}) + \frac{\partial}{\partial x}(\vec{\pi}_x, \vec{v}) + \frac{\partial}{\partial y}(\vec{\pi}_y, \vec{v}) + \frac{\partial}{\partial z}(\vec{\pi}_z, \vec{v}) - div\vec{q} + \rho Q,$$

in which strains may be taken in the form of (29),(32) or transform (35) into the below form

$$\rho c_v(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}) = (\vec{\pi}_x, \frac{\partial \vec{v}}{\partial x}) +$$

$$+ (\vec{\pi}_y, \frac{\partial \vec{v}}{\partial y}) + (\vec{\pi}_z, \frac{\partial \vec{v}}{\partial z}) + div(\lambda gradT) + \rho Q \qquad (36)$$

In incompressible fluids $\rho = const$, if viscosity is constant, $\mu = const$. For many laminar flows friction coefficient $k = 0$.

In turbulent flows friction is caused because of mutual penetration of particles from neighbouring layers, which is eventually connected with velocity fluctuations in *Euler* approach. Therefore, in the system (33) coefficient $''k''$ must be connected with the degree of turbulence in the given point of flow and, most probably, was selected in the form

$$k = k_0(u'^2 + v'^2 + w'^2)^{\frac{1}{2}} / w_{масш}, \quad k_0 = const > 0, \qquad (37)$$

where $u' = u - \bar{u}, v' = v - \bar{v}, w' = w - \bar{w}$ -fluctuations, $\bar{u}, \bar{v}, \bar{w}$ - average in the given point values of velocity components. First alternative – as $\bar{u}, \bar{v}, \bar{w}$ may be used averaged for the previous time interval values. Second option – use of net averaging at the given moment of time in a definite net node with indexes $i, j, m,$ for instance, arithmetical mean value of function

$$\bar{f}_{ijm}^{n} = (f_{i-1jm}^{n} + f_{i+1jm}^{n} + f_{ijm}^{n} + f_{ij-1m}^{n} + f_{ij+1m}^{n} + f_{ijn-1}^{n} + f_{ijn+1}^{n})/7, \quad (38)$$

where n - number of time layer t_n (Of course, for averaging (38) it is possible to involve great number of surrounding nodes). Then fluctuations at the given moment of time t_n are easily counted

$$u_{lmn}'^{n} = u_{lmn}^{n} - \bar{u}_{lmn}^{n}, \; v_{lmn}'^{n} = v_{lmn}^{n} - \bar{v}_{lmn}^{n}, \quad w_{lmn}'^{n} = w_{lmn}^{n} - \bar{w}_{lmn}^{n}, \qquad (39)$$

where $u_{lmn}^{n}, v_{lmn}^{n}, w_{lmn}^{n}$ – actual values of velocity components, counted by difference scheme on computer. Third option – is to assume constancy $k = const$ and varying it, to study obtained numerical results. Fourth alternative – is to determine values $k = const$, comparing results of numerical computations of flows for longitudinal flow around disc or in pipe with experimental data.

Among these three alternatives, the most hardly executable is the first one, where fluctuations u', v', w' must be calculated from appropriate system of equations, methodology of obtaining of which is developed in §3. Obviously in second and third options, involvement of fluctuations equations is not required.

While setting initial and boundary conditions for computation of turbulent flows, random disturbances must be included /2/.

Numerical solution of initially-boundary value problem for equations (33) and other is performed on the basis of difference schemes /4/.

Note. In those points of flow (in difference methods – in net nodes), where velocity components are equal to 0: ***u=0,v=0,w=0*** in (32) and in equations (33) cosines

$$u(u^2+w^2)^{-\frac{1}{2}}, v(v^2+w^2)^{-\frac{1}{2}}, u(u^2+v^2)^{-\frac{1}{2}},$$
$$w(v^2+w^2)^{-\frac{1}{2}}, v(u^2+v^2)^{-\frac{1}{2}}, w(u^2+w^2)^{-\frac{1}{2}},$$

will become a equivocation $\frac{0}{0}$. In this case, basing on chaotic nature of turbulent flows, these cosines may be taken with the help of sensor (built-in procedure) of random numbers as equal to random

value from interval [-1,1], if such necessity is confirmed by L'Hospital rule.

Note. Due to arbitrariness of selection of the second viscosity coefficient, if to include $\mu' = \frac{1}{3}\mu$, the given equations will reduce by not less than 40 derivatives.

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Chapter 4. FALLACY OF APPLYING "*DARCY law*" IN FILTERING THEORY

Polubarinova-Kochina P.Ya. in /1/ certify that engineer *Darcy* in 1856 published in /2/ formula of linear dependence of filtration velocity $v = |\vec{v}|$ from piezometric gradient *s:*

$$v = -\chi \frac{dh}{ds}, \qquad (1)$$

referred to in /1/ *the Darcy law.* In (1) $h = \frac{p}{\rho g} + z$ is declared a piezometric head, axis *z* is directed straight up, direction *s* makes angle of piezometric gradient with axis *z*, χ - filtration factor (for sand χ is measured in interval *0.01- 0.0001 m/sec*). Important is that

in (1) $v = \sqrt{v_x^2 + v_y^2 + v_z^2} = |\vec{v}|$, since due to *porosity of medium,* filtration is essential spatial motion, occurring with velocity $\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$, due to which, most probably, Darcy measured in his experiments not only one velocity component, but in particular full speed value $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$, therefore, formula (1) must be represented in details as follows:

$$\sqrt{v_x^2 + v_y^2 + v_z^2} = -\chi \frac{dh}{ds} \qquad (2)$$

In the event when direction of gradient is vertical, i.e. *s=z,* from (2) results, as a consequence, the following formula:

$$\sqrt{v_x^2 + v_y^2 + v_z^2} = -\chi \frac{d}{dz}\left(\frac{p}{\rho g} + z\right) \qquad (3)$$

Again porosity of medium, *fluid particles going around solid components of porous medium*, makes the flow three-dimensional: $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$, which is accounted for in (3).

In the same book /1/ values of filtration factor χ for various media are given. It can be expected that values of filtration factor $\bar{\chi}$ will be different, if following second *Newton* law, according to wchi ***force*** causes ***acceleration*,** i.e. instead of *Darcy* formula, to use the main law of dynamics

$$\frac{dv_s}{dt} = -\bar{\chi}\frac{dh}{ds}, \quad v_s = (\vec{v}, \vec{s}), \quad |\vec{s}| = 1,$$

and thus avoid contradictions with laws of physics. In principle it is necessary to base on the fact that in building mathematical models of this or that phenomenon, **where *forces causing motion act*, to indisputably follow *Newton* laws, laws of conservation of matter and energy. Models, violating these laws must be rejected, as having no physical sense.**

§1. How "*Darcy law*" was created for multidimensional filtration

Darcy formula (2), transformed to formula (3) of vertical gradient, further was *reduced to* formula

$$w = -\chi \frac{d}{dz}(\frac{p}{\rho g} + z),$$

introduced only for one vertical component of velocity *w*, generalized for spatial filtering with velocity $\vec{v} = u\vec{i} + v\vec{j} + w\vec{k}$ and represented in ***vector form as "Darcy law"*** /3/:

$$\vec{v} = -\chi grad(\frac{p}{\rho g} + z) \qquad (4)$$

It is necessary to note here that the basis for (1) *Darcy* laid a number of experimental data, therefore use of the above-presented formula in engineering practice for rough estimates of vertical fluid flow rate in soil is quite acceptable, whereas generalization (4) conflicts laws of physics and encounters a number of **problems of setting boundary layers adequate to the process of filtration.**

Later on an effort was made to give justification to "*Darcy law*" (4), using vicous media dynamics equation

$$\rho \frac{d\vec{v}}{dt} = -gradp + \rho\vec{F} + \mu\Delta\vec{v}, \qquad (5)$$

in which acceleration is equaled to zero

$$\frac{d\vec{v}}{dt} = 0, \qquad (6)$$

and, taking into account essential resistance of medium to fluid motion (filtration), was made an ***assumption***, that in soil consisting of solid particles it is possible to use formula of friction in solid bodies slipping (see *Polubarinova-Kochina P.Ya.* /1/)

$$\vec{F}_{mp} = k\vec{v},$$

according to which, forces of viscous friction are assumed to be proportional to velocity

$$\mu\Delta\vec{v} = -k\vec{v}$$

In a result, "***Darcy law***" for multidimensional filtration is applied in the form of a system

$$k\vec{v} = -gradp + \rho\vec{F}, \qquad (7)$$

$$div\vec{v} = 0, \qquad (8)$$

$\vec{F}$ - density of bulk forces.

Equation (5) is the analogue for the second *Newton* law, falling on a unit of volume

$$\rho(\frac{\partial\vec{v}}{\partial t}+u\frac{\partial\vec{v}}{\partial x}+v\frac{\partial\vec{v}}{\partial y}+w\frac{\partial\vec{v}}{\partial z})=-gradp+\rho\vec{F}+\mu\Delta\vec{v}\ ,$$

according to which in model (7), (8) it ocurrs that during filtration of fluid particles acceleration

$$\frac{d\vec{v}}{dt}\equiv\frac{\partial\vec{v}}{\partial t}+u\frac{\partial\vec{v}}{\partial x}+v\frac{\partial\vec{v}}{\partial y}+w\frac{\partial\vec{v}}{\partial z}=0 \qquad (8)$$

are equal to zero, accordingly, their velocities $\vec{v}=const$ are constants and according to the first *Newton* law fluid particles must move along ***rectilinear*** **trajectories.**

§2. Problem of setting boundary conditions

Complexity of setting boundary conditions adequate to simulated physical process of filtration is already seen from one-dimensional equations of "*Darcy law*". Indeed, system of equations (7)-(8)

$$k\vec{v}=-gradp+\rho\vec{F}\ ,\ div\vec{v}=0$$

in one-dimensional flow in parallel to axis z with velocity $\vec{v}=0\vec{i}+0\vec{j}+w\vec{k}$ in projection is reduced to equations

$$kw=-\frac{dp}{dz}-\rho g,\ \ \frac{dw}{dz}=0$$

From the last equation results constancy of velocity $w=const$. For defining this *const*, **it is necessary and enough to set velocity in any one point** $w(z_0)=w_0=const$. Then, **for computation of pressure from the first equation**

$$kw_0=-\frac{dp}{dz}-\rho g$$

it is also enough to set its value in one point and solution will have the following form

$$p(z)=p(z_0)-(kw_0+\rho g)(z-z_0)$$

This circumstance must be taken into account in the theory of

multidimensional filtration while formulating boundary conditions for components of velocity and pressure.

§3. On setting boundary conditions for unknown functions *u, v, w, p*

Here arises a number of problems while setting boundary conditions for equations in partial derivatives of "*Darcy law*", related to fact that in (7), (8) included are only ***first derivatives*** from unknown functions *u, v, w, p*. Indeed, in Cartesian system these equations have the following form

$$ku = -\frac{\partial p}{\partial x} + \rho F_x, kv = -\frac{\partial p}{\partial y} + \rho F_y, kw = -\frac{\partial p}{\partial z} + \rho F_z \qquad (9)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad , \qquad (10)$$

accordingly, for ***u*** and ***p*** there must be set ***one boundary condition*** on lines parallel to coordinate axis ***x***, for ***v*** and ***p - one boundary condition*** on lines parallel to coordinate axis ***y***, for ***w*** and ***p - one boundary condition*** on lines parallel to coordinate axis ***z*** , *i.e. only in separate sites of boundary of flow area under study*, ***and not on the whole boundary as a whole.***

According to theorem of *Ostrogradsky-Gauss* $\iiint\limits_{\tau} div\vec{v}d\tau = \oint\!\!\!\!\iint\limits_{\sigma}(\vec{v},\vec{n})d\sigma$ from continuity equation (8) results a condition for boundary values of velocity vector components

$$\oint\!\!\!\!\iint\limits_{\sigma}(\vec{v},\vec{n})d\sigma = 0 \qquad (11)$$

§4. Boundary conditions of *Dirichlet* and *von Neumann*

Problems of setting boundary conditions arises for two-dimensional equations

$$ku = -\frac{\partial p}{\partial x} + \rho F_x, kw = -\frac{\partial p}{\partial z} + \rho F_z \qquad (12)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad (13)$$

For simplicity of reasoning, we'll select boundary σ of plain domain τ in the form of a rectangle. According to above-said, if on section of boundary σ_1 set was boundary condition $u\big|_{\sigma_1} = \varphi_1$, and on section of boundary σ_2 set was $w\big|_{\sigma_2} = \varphi_2$, then due to contained in equations (12) first derivatives of pressure $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial z}$ there must be set boundary conditions on opposite sections of boundary σ_3, σ_4 ***in the form p=*** λ and ***p=q.***

Used until now methodology of solving equations (7),(8) uses elliptic equation

$$div(\frac{1}{k}(gradp - \rho\vec{F})) = 0 \ , \qquad (14)$$

which is the result of substituting (7) into (8). Integral for area τ from both parts of (14) leads in the general case to equality

$$\iiint\limits_{\tau} div(\frac{1}{k}(gradp - \rho\vec{F}))d\tau = \oint\!\!\!\oint\limits_{\sigma}(\frac{1}{k}(gradp - \rho\vec{F}),\vec{n})d\sigma = 0 \qquad (14')$$

Thus, for two-dimensional equation (14) due to association (12) boundary conditions will be

$$(-\frac{\partial p}{\partial x} + \rho F_x)\Big|_{\sigma_1} = k\varphi_1, (-\frac{\partial p}{\partial z} + \rho F_z)\Big|_{\sigma_2} = k\varphi_2 \quad ,$$

$$p\Big|_{\sigma_3} = q, \ p\Big|_{\sigma_4} = \sigma \qquad (15)$$

Having solved boundary value problem (14), (15) in relation to pressure ***p***, velocity components are calculated from formulae (12):

$$u = (-\frac{\partial p}{\partial x} + \rho F_x) / k, w = (-\frac{\partial p}{\partial z} + \rho F_z) / k \qquad (16)$$

throughout the area τ, including sections σ_3, σ_4 of the boundary.

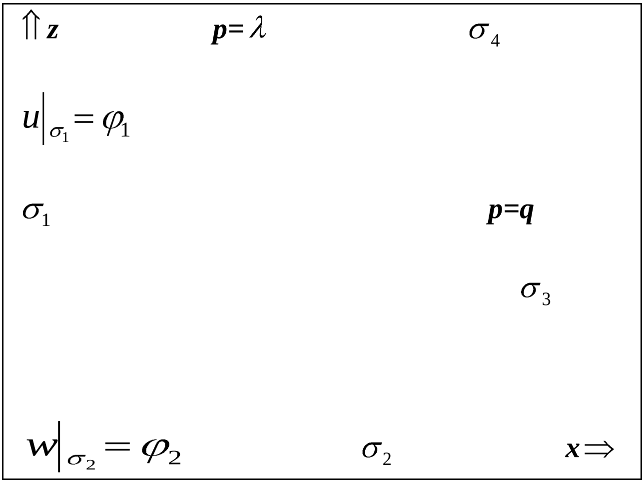

Fig.1

The question arises: do values of velocity components obtained in the result of boundary value problem (14),(15) solution, satisfy condition (11), which for plain domain τ will take a form

$$\oint_{\sigma} (\vec{v}, \vec{n}) d\sigma = 0 \ , \qquad (17)$$

in this case σ - curve limiting τ .

§5. Boundary conditions of *von Neumann* type

On the same two-dimensional problem, let's consider setting *boundary conditions of von Neumann* on sections σ_3 and σ_4 of the boundary, on σ_1 and σ_2 they are already set:

$$(-\frac{\partial p}{\partial x} + \rho F_x)\Big|_{\sigma_1} = k\varphi_1, (-\frac{\partial p}{\partial z} + \rho F_z)\Big|_{\sigma_2} = k\varphi_2, \qquad (18')$$

$$(\frac{\partial p}{\partial x})\Big|_{\sigma_3} = \varphi_3, (\frac{\partial p}{\partial z})\Big|_{\sigma_4} = \varphi_4 \qquad (18)$$

According to *Darcy law* (7), boundary conditions (18) result in the equality of velocity boundaries on *sections* σ_3 and σ_4

$$u\Big|_{\sigma_3} = (-\varphi_3 + \rho F_x)/k \ , \quad w\Big|_{\sigma_4} = (-\varphi_4 + \rho F_z)/k \qquad (19)$$

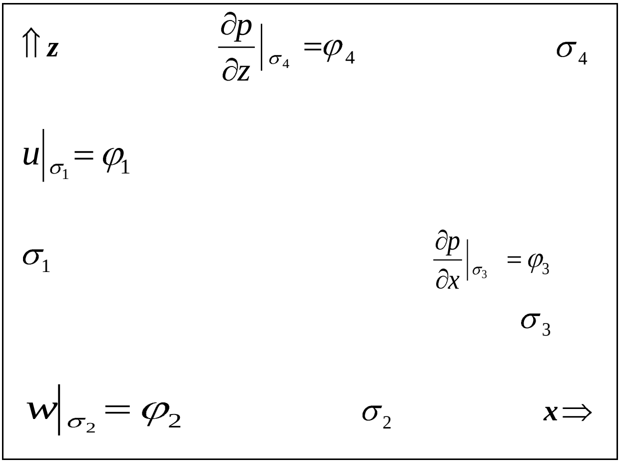

Fig.2

Thus, boundary values

$$\varphi_1, \varphi_2, (-\varphi_3 + \rho F_x)/k, (-\varphi_4 + \rho F_z)/k$$

must be satisfying ***condition (17), and in three-dimensional problem, condition (11). There must be satisfied solvability condition (14') with account to boundary conditions (18), (18').***

Paradox of application of "*Darcy law*" in form of multidimensional problem (7),(8) is in the fact that due to continuity equation (13) for ***u*** one boundary condition $u\big|_{\sigma_1} = \varphi_1$ will be enough, and because of (19) there arises ***the second boundary condition*** $u\big|_{\sigma_3} = (-\varphi_3 + \rho F_x)/k$, which is needless, since equation (13) of the first order by ***x***, similar situation ocurrs also for ***w***: two boundary conditions

$$w\big|_{\sigma_2} = \varphi_2 \ , \ w\big|_{\sigma_4} = (-\varphi_4 + \rho F_z)/k \ ,$$

because in continuity equation (13) there is first derivative $\frac{\partial w}{\partial z}$, and the same situation of excess boundary conditions for velocity components arises in three-domensional problems.

It is clear that boundary condition of *von Neumann*

$$\frac{\partial p}{\partial n}\Big|_{\sigma} = (gradp, \vec{n})\Big|_{\sigma} = \psi$$

must satisfy according to (14’) on area boundary

$$\oiint_{\sigma} (\frac{1}{k}(gradp - \rho\vec{F}), \vec{n}) d\sigma = \oiint_{\sigma} (\frac{1}{k}\psi - \frac{1}{k}(\rho\vec{F}, \vec{n})) d\sigma = 0$$

solvability criterion $\oiint_{\sigma} (\frac{1}{k}\psi) d\sigma = \oiint_{\sigma} (\frac{1}{k}(\rho\vec{F}, \vec{n})) d\sigma$

§6. **Paradoxes** of **“*Darcy law*” (7), (8)**

First paradox: **“*Darcy law*” (7), (8) in conservative field of force describeees vortex-free, i.e. potential flow.**

Let’s compute circulation of velocity vector $\oint_{\Gamma} (\vec{v}, d\vec{r})$ by arbitrary shape Γ, which in filtering theory by “***Darcy law*” (7), (8)** is equal to:

$$\oint_{\Gamma} (\vec{v}, d\vec{r}) = \oint_{\Gamma} ((-gradp + \rho\vec{F})/k, d\vec{r}) \qquad (20)$$

Let the flow ake place under action of conservative gravity

$$\vec{F} = -grad\Pi \ , \qquad (21)$$

where Π – potential energy.

In this case, integrand in (20) is equal to

$$\begin{aligned} &((-gradp + \rho\vec{F})/k, d\vec{r}) = ((-gradp/\rho - grad\Pi)\rho/k, d\vec{r}) = \\ &= -(\rho/k) d(p/\rho + \Pi) \end{aligned} \qquad (22)$$

Due to this for $\rho = const$, $k=const$ integral in the right part (20) will be equal to zero

$$\oint_{\Gamma} (\vec{v}, d\vec{r}) = \oint_{\Gamma} ((-gradp + \rho\vec{F})/k, d\vec{r}) \ = -(\rho/k)\oint_{\Gamma} d(p/\rho + \Pi) = 0 \quad (23)$$

As the result from (20) for potential forces obtained is equality to zero for circulation of velocity vector within any closed shape

$$\oint_{\Gamma} (\vec{v}, d\vec{r}) = 0 \qquad (24)$$

According to *Stokes* theorem, there is transition to surface integral

$$\oint_{\Gamma} (\vec{v}, d\vec{r}) = \iint_{S} (rot\vec{v}, \vec{n}) ds = 0 \qquad (25)$$

Here *S* – arbitrary surface, stretched to shape *Г*. By *Du Bois - Raymond* lemma, from (25) results equality to zero of velocity vortex (vorticity)

$$rot\vec{v} = 0 \qquad (26)$$

throughout the area of flow.

Another proof of equality (26) results from identity

$$rot\vec{v} = -\frac{1}{k} rotgrad\,(p + \rho\Pi) \equiv 0$$

Flows, in which velocity vortex is everywhere equal to zero $rot\vec{v} = 0$, are, as is known, *potential floes of ideal fluids* /3/. Velocity potential for two-dimensional flows was considered by *Polubarinova – Kochina P.Ya.* in /1/. In the same book /1/ it is admitted that resistance forces, which affect fluid particle in a pore, **depend on internal friction of fluid. Internal friction – is the integral feature of viscous fluids, in viscous fluids potential flows are impossible, vorticity is different from zero** $rot\vec{v} \neq 0$**.** Thus, filtration flow by “*Darcy law*” is initially *irrotational flow of ideal fluid*, in ideal fluids there are no friction force, no moleculartransport, which **conflicts reality**, in particular when extra-heavy oil motion is studied.

Second paradox: **“*Darcy law*” (7), (8) contradicts to law of energy conservation.**

Velocity potential *Ф* is introduced by formula

$$\vec{v} = grad\Phi \qquad (27)$$

Substituting (21) and (27) into equation (7) of *Darcy law*, we’ll find

$$grad\,(k\Phi + p + \rho\Pi) = 0\ , \qquad (28)$$

which results in integral

$$k\Phi + p + \rho\Pi = const\ , \qquad (29)$$

where *const* is the same for the whole flow area. On the other part, for steady-state irrotational flows of ideal fluids there is the integral of *Lagrange-Cauchy* /3/.

$$\rho\frac{|\vec{v}|^2}{2}+p+\rho\Pi=const \qquad (30)$$

Obviously, (29) does not result in integral of *Lagrange-Cauchy*. Equating *const* in (29) and (30), we arrive to paradoxical equality for *"Darcy law"*

$$k\Phi=\rho\frac{|grad\,\Phi|^2}{2} \qquad (31)$$

Paradoxicality of (31) is connected with the fact that velocity potential Φ is determined accurate to arbitrary constant /2/, and gradient Φ neglects this constant.

It is easy to see that integral of *Lagrange-Cauchy* (30) expresses law of conservation of mechanical energy in the unit of continuum volume.

And in flows by "*Darcy law*" appears integral **(29). Obviously, integral (29) of** *"Darcy law"* **conflicts with law of energy conservation (31).**

Third paradox: **"*Darcy law*" (7),(8) conflicts with law of variation of momentum or second *Newton* law.**

Major dynamics law $\frac{d(m\vec{v})}{dt}=\sum_k \vec{f}_k$ states that any force $\sum_k \vec{f}_k$ causes variation of momentum $m\vec{v}$, therefore is formulated as law of momentum variation. Having differentiated, it is possible to present it in the form

$$m\frac{d\vec{v}}{dt}+\vec{v}\frac{dm}{dt}=\sum_k \vec{f}_k$$

Any motion of material medium in *Newton* mechanics is subordinated to this law, accordingly, motion in porous medium arranged in the form of "*Darcy law*"

$$k\vec{v}=-gradp+\rho\vec{F},$$

must follow this law. Simple comparison of these two laws, one of which is pseudo "*Darcy law*" leads to the following fact

$$\sum_k \vec{f}_k = -\,gradp + \rho\vec{F}\ ,\ \vec{v}\frac{dm}{dt} = k\vec{v}\ ,\ m\frac{d\vec{v}}{dt} = 0$$

From the last equality it results that in pseudo- "*Darcy law*" **acceleration is equal to zero** $\frac{d\vec{v}}{dt} = 0$.

But his means that according to 1st *Newton* law, particles must accomplish straight line motion with constant velocities, or rest, which is possible only in one-dimensional flow towards axis "z".

In hydrodynamics, descriptive illustration of such flows are: *Couette* flow, channel *Poiseuille* flows between parallel planes, *Hagen–Poiseuille* flow in circular pipe etc. /3/.

Due to this application of *"Darcy law" (7),(8) for computation of two-dimensional and three-dimensional flows at once loses sense because of contradiction with 1st Newton law, as soon as curved streamlines appear in the flow. There can be no rectilinearity of streamlines due to above-given paradoxes with boundary conditions.*

Equating to zero of acceleration $\frac{\partial\vec{v}}{\partial t} + u\frac{\partial\vec{v}}{\partial x} + v\frac{\partial\vec{v}}{\partial y} + w\frac{\partial\vec{v}}{\partial z} = 0$ is justified by the following considerations. First: in steady-state flow, of course, partial time derivative $\frac{\partial\vec{v}}{\partial t} = 0$, but for nonsteady flow local acceleration is not equal to zero $\frac{\partial\vec{v}}{\partial t} \neq 0$, and undertaken in monograph /1/ effort to equate it to zero ca not stand stricture. Second: spatial transposition members $u\frac{\partial\vec{v}}{\partial x} + v\frac{\partial\vec{v}}{\partial y} + u\frac{\partial\vec{v}}{\partial z}$ are negligibly small. Steady-state of filtration, does not probably always take place due to action of external conditions, and products in $u\frac{\partial\vec{v}}{\partial x} + v\frac{\partial\vec{v}}{\partial y} + w\frac{\partial\vec{v}}{\partial z}$ may be not small at all. Let's show this on *analogue* of only one product from thus expression of transition, for example, on $w\frac{dw}{dz}$. Let $w = \varepsilon F(\cos\alpha z)$, where $\alpha = const >> 1$ is

very large but finite number, ε - parameter, providing infinitesimality of unction *w, F*- limited with derivative differentiable function of its argument, then derivative will be equal to $\frac{dw}{dz} = -\varepsilon F'\alpha \sin \alpha z$. ***Therefore, product*** $w\frac{\partial w}{\partial z} = -\alpha \varepsilon^2 FF' \sin \alpha z$ is the ***finite number, it is enough to include*** $\alpha = const * \varepsilon^{-2}$**.** Sufficient number of similar examples may be given, for example, functions with weak discontinuity, in order to prove that ***equating to zero of acceleration***

$$\frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z} = 0$$

is unjustified. **Unjustified, first of all, because of contradiction with the major dynamics law – in particular, with 2nd *Newton* law, which says: action of force results in acceleration, not equal to zero:**

$$\frac{d\vec{v}}{dt} \neq 0$$

(Paradox – is it real that fountains of oil and water from underground occur under negligibly small accelerations?)

As was noted in (14), widely spread method of solving equations (7), (8) of "*Darcy law*" is reduction of this system to one elliptical equation

$$div(\frac{1}{k} gradp) = div(\frac{1}{k} \rho\vec{F})$$

in above-indicated boundary conditions.

It is necessary to admit that ***the most important criterion of accuracy*** of obtained solution of equation (14) is that for velocity vector $\vec{v}$, defined by formula (7) *"Darcy law"*, not only condition (11) must be exercised, but first of all equality to zero of acceleration

$$\frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z} = 0 \ ,$$

i.e. pressure in steady-state filtration must satisfy equation resulted from the below equality:

$$(-\frac{\partial p}{\partial x}+\rho F_x)\frac{\partial}{\partial x}((-gradp+\rho\vec{F})/k)+$$
$$+(-\frac{\partial p}{\partial y}+\rho F_y)\frac{\partial}{\partial y}((-gradp+\rho\vec{F})/k)+$$
$$+(-\frac{\partial p}{\partial z}+\rho F_z)\frac{\partial}{\partial z}((-gradp+\rho\vec{F})/k)=0$$

in all points of investigated area, which is not an automatic consequence of solution of equations (7), (8), as this occurs in flows of *Couette*, *Poiseuille* between parallel planes, *Hagen–Poiseuille* in circular pipe, where acceleration is identically zero /3/.

Obtained is 4th paradox: pressure in *"Darcy law"(7),(8)* is defined through solution of equation (14), on the one part, according to the first and second *Newton* laws, pressure must be also solution of the system of 3 equations:

$$(\rho F_x-\frac{\partial p}{\partial x})\frac{\partial}{\partial x}[(\rho F_x-\frac{\partial p}{\partial x})/k]+(\rho F_y-\frac{\partial p}{\partial y})\frac{\partial}{\partial y}[(\rho F_x-\frac{\partial p}{\partial x})/k]+$$
$$+(\rho F_z-\frac{\partial p}{\partial z})\frac{\partial}{\partial z}[(\rho F_x-\frac{\partial p}{\partial x})/k]=0,$$
$$(\rho F_x-\frac{\partial p}{\partial x})\frac{\partial}{\partial x}[(\rho F_y-\frac{\partial p}{\partial y})/k]+(\rho F_y-\frac{\partial p}{\partial y})\frac{\partial}{\partial y}[(\rho F_y-\frac{\partial p}{\partial y})/k]+$$
$$+(\rho F_z-\frac{\partial p}{\partial z})\frac{\partial}{\partial z}[(\rho F_y-\frac{\partial p}{\partial y})/k]=0,$$
$$(\rho F_x-\frac{\partial p}{\partial x})\frac{\partial}{\partial x}[(\rho F_z-\frac{\partial p}{\partial z})/k]+(\rho F_y-\frac{\partial p}{\partial y})\frac{\partial}{\partial y}[(\rho F_z-\frac{\partial p}{\partial z})/k]+$$
$$+(\rho F_z-\frac{\partial p}{\partial z})\frac{\partial}{\partial z}[(\rho F_z-\frac{\partial p}{\partial z})/k]=0$$

Alltogether, for one pressure function *p* arise 4 equations which once again shows ***irrationality*** of applying *"Darcy law" (7),(8)* in the theory of multidimensional filtration.

Due to the above-mentioned reasons, application of so-called *"Darcy law"* **in the form of system of equations (7),(8) in the filtration theory is erroneous.**

§7. In «*Darcy law*» only in one-dimensional steady-state flow, all three laws of physics are observed, *if* equation of continuity accounts for mass source

Let's consider one-dimensional steady-state flow of ideal fluid with velocity components $u=0, v=0, w\neq 0,$ parallel to direction of gravity $\vec{g}$:

$$\rho w \frac{dw}{dz} = -\frac{dp}{dz} - \rho g \tag{32}$$

This equation is projection to axis "*z" of Euler equation*

$$\rho \frac{d\vec{v}}{dt} = -gradp + \rho\vec{F}$$

Multiplying both parts of this equation by elementary volume $\delta\tau$, we'll arrive to the second *Newton* law

$$\delta m \frac{d\vec{v}}{dt} = \vec{f} \ ,$$

where $\vec{f} = (-gradp + \rho\vec{F})\delta\tau$ - resultant force, $\delta m = \rho\delta\tau$ - mass.

Further on, taking into consideration filtration in soil, in the law of mass conservation let's introduce strength of source or drain in the following form

$$\rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = k \tag{33}$$

This equation in case of one-dimensional flow $u=0, v=0, w\neq 0,$ is reduced to the form

$$\rho \frac{dw}{dz} = k \tag{34}$$

Projection of acceleration

$$\frac{\partial\vec{v}}{\partial t} + u\frac{\partial\vec{v}}{\partial x} + v\frac{\partial\vec{v}}{\partial y} + w\frac{\partial\vec{v}}{\partial z}$$

for steady-state flow towards ***z*** will be equal to

$$w\frac{dw}{dz} = w\frac{k}{\rho}$$

Substituting (34) into (32)

$$kw = -\frac{dp}{dz} - \rho g \tag{35}$$

we'll obtain formula (3). In conservative field of gravity force potential energy is equal to $\Pi = gz + const$, due to which law of energy conservation (30) (integral of *Lagrange*-*Cauchy*) for one-dimensional flow takes the form:

$$\rho\frac{w^2}{2} + p + \rho gz = const\ , \tag{36}$$

having differentiated which on ***z* we obtain dynamics equation (32).**

On the other part equation (32) is brought to equivalent form

$$\frac{d}{dz}(\frac{\rho w^2}{2} + p + \rho gz) = 0$$

from which obviously results law of conservation of mechanical energy (36) in unit of volume of medium.

Clear is the following simple circumstance – according to formula (1) of engineer *Darcy*, first problem of dynamics is solved: knowing velocity of fluid $w(z_0)$ and pressure $p_0 = p(z_0)$ on the given level, it is possible to compute force affecting surface at the distance of $(z - z_0)$ from the selected level. Let's demonstrate it on one-dimensional flow. From (35) it is easy to derive

$$w(z_0) = -\frac{1}{k}(\frac{p - p_0}{z - z_0} + \rho g),$$

wherefrom pressure on level *z* is computed*:*

$$p = p_0 - (z - z_0)[kw(z_0) + \rho g],$$

where in the right part, there are measured quantities. Preliminary, for calculation of coefficient *k* formula (34) *is applied*.

Integrating (34) by height of filtration, it is easy to establish

$$\rho(w(z_2) - w(z_1)) = k(z_2 - z_1)$$

wherefrom the coefficient is calculated

$$k = \frac{\rho(w(z_2) - w(z_1))}{z_2 - z_1},$$

where, obviously, levels z_1 *and* z_2 are defined in experimental soil sample to be so, that on these levels velocities $w(z_2)$, $w(z_1)$ are known.

Thus, founders of so-called *"Darcy law"* instead of the first problem of dynamics are trying to simultaneously solve the first and second problems of dynamics, which in the result led to above-mentioned paradoxes.

§8. Forzheimer model of filtration in saturated porous media

As opposed to "*Darcy law*" more suitable (from the point of view of observance of laws of physics) for describing filtration in saturated porous media is the *Forzheimer* model /5/

$$\rho_f \frac{\partial \vec{v}}{\partial t} = -\varphi(\nabla(p + \rho_f gz) + \rho_f \frac{\nu}{K}\vec{v} + \rho_f c_f K^{-\frac{1}{2}} |\vec{v}|\vec{v}), \qquad (37)$$

$$div\vec{v} = 0 \qquad (38)$$

These equations take into account **local acceleration,** equation (37) is represented as major dynamics equation, **thus, observed is second *Newton* law,** as opposed to "*Darcy law*". In (37) φ - coefficient of medium porosity, K – permeability coefficient, ρ_f - fluid density, c_f - dimensionless *Forzheimer* friction coefficient. Accounted is also fluid viscosity, since kinematic viscosity coefficient ν is included.

Regretfully, for the system of *Forzheimer* arises similar to the above-mentioned problem of setting boundary conditions for pressure and velocity vector, and adequacy of obtained solution to simulated physical process depends on boundary conditions.

§9. *Forzheimer* model correction

The indicated problem is easily removed, if *Forzheimer* model is altered, making it close to equations of viscous fluid, at the same time replacing **local acceleration for an individual one**

$$\rho_f \frac{d\vec{v}}{dt} = -\varphi[\nabla(p+\rho_f gz)+\rho_f \frac{\nu}{K}\vec{v}+\rho_f c_f K^{-\frac{1}{2}}|\vec{v}|\vec{v}]+\mu\Delta\vec{v},$$
$$div\vec{v} = 0 \qquad (39)$$

Numerical solution of initially-boundary value problem for system (39) is carried out by methods developed for viscous fluids equations /6/.

§10. *Numerov* model of filtration in saturated porous media

Numerov S.N. in 1968 turned attention to necessity of taking account of inertia forces in major equations of filtering theory and proposed to use in /8/ the following system of equations:

$$\frac{1}{g\sigma}\frac{\partial\vec{v}}{\partial t}+\frac{1}{g\sigma^2}(\vec{v},\nabla)\vec{v}+gradh+f(\vec{v})\vec{v}=0,\ div\vec{v}=0$$

Thus, *Numerov S.N.* for the first time drew attention of researchers to inadmissibility of violating *Newton* laws of dynamics, which state that: “mass by acceleration is equal to force”, if acceleration is equal to zero, a body performs straight line motion with constant velocity $\vec{v}=const$ or is resting.

§11. Interest to “*Darcy law*” is related to applications of complex variable theory

Indeed, in plane problems (7)-(8) elliptic equation is obtained for pressure, if $\rho\vec{F}=const$:

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=0, \qquad (40)$$

for finding solution of which effectively used was theory of complex variable functions, which is in sufficient details described in /1/,/4/ etc. At present, development of computing machinery and mathematics allow obtaining of numerical solution of equation (40) with the given boundary conditions in a matter of seconds, therefore equation (40) is of interest more as learning material /7/, than scientific and research, due to above-mentioned reasons.

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Chapter 5. FALLACY OF THE “FICTITIOUS DOMAIN METHOD”

Recently, there appeared many works devoted to *fictitious domain method* (f.d.m.), idea of which was in due time proposed by *V.K. Saulyev.*

Fallacy of “f.d.m.” is that solutions of two independent initially-boundary value problems for differential equations in partial derivatives are connected in arbitrary manner. There are especially many works on “f.d.m.” for *Navier-Stokes* equations.

Idea of “f.d.m.” is as follows. Let in “inconvenient” for introducing computational meshes of nonrectangular physical region *D* with boundary *S*, initially-boundary value problem for *Navier-Stokes* equations is solved

$$\rho[\frac{\partial\vec{v}}{\partial t}+(\vec{v},\nabla)\vec{v}]+\nabla p=\mu\Delta\vec{v}+\rho\vec{F},(\nabla,\vec{v})=0,\vec{v}\mid_{t=0}=\vec{d},\vec{v}\mid_S=\vec{\varphi} \quad (1)$$

In "f.d.m." to regions *D* added is *fictitious domain* D_ε with boundary S_ε, in which initially-boundary value problem is solved, for example, for case $\vec{\varphi}_\varepsilon=0$ like this:

$$\rho[\frac{\partial\vec{v}_\varepsilon}{\partial t}+(\vec{v}_\varepsilon,\nabla)\vec{v}_\varepsilon]+\nabla p_\varepsilon=\mu\Delta\vec{v}_\varepsilon+\rho\vec{F}_\varepsilon-\frac{\vec{v}_\varepsilon}{\varepsilon},(\nabla,\vec{v}_\varepsilon)=0,\vec{v}_\varepsilon\mid_{t=0}=\vec{d}_\varepsilon,\vec{v}_\varepsilon\mid_S=\vec{\varphi}_\varepsilon \quad (2)$$

It is supposed that region $D\bigcup D_\varepsilon$ is convenient for introducing net domain while solving with the help of finite-difference methods.

Here, it is necessary to pay special attention to *additive term to equation* (2) $\frac{\vec{v}_\varepsilon}{\varepsilon}$, in denominator of which, artificially introduced parameter ε tents to zero: $\varepsilon\to 0$!

It would seem that a single look at equation (2) is enough to reject f.d.m. Obviously, ε - equation (2) absolutely does not coinside with *Navier-Stokes* equation (1) with small parameter $\varepsilon\to 0$, but tents to it under conversely large parameter value $\varepsilon\to\infty$. And the following questions arise at once.

***Question* 1**. What must be specific numerical value of dimensional parameter ε (dimensionality [ε]=kg/($м^3$c))? What are criteria for its selection?

It cannot be equal to zero. Assuming that ε =0, *fictitious* problem (2) transforms into a nugatory equality

$$\vec{v}_\varepsilon=0 \quad (3)$$

throughout the whole *fictitious* domain D_ε, which means there will be no necessity in initially-boundary ε - problem (2). At the same

time it is obvious that continuation of solution of initial *real* problem (1) into fictitious domain D_ε will be non-zero

$$\vec{v}\Big|_{D_\varepsilon} \neq 0 \quad , \tag{4}$$

i.e. there is a contradiction with (3).

***Question* 2.** If $\varepsilon \neq 0$, what boundary conditions must meet velocity vector $\vec{v}_\varepsilon\Big|_{S_\varepsilon} = \vec{\varphi}_\varepsilon$ on boundary S_ε, i.e. how must be selected vector $\vec{\varphi}_\varepsilon$ so that solution of *fictitious* ε-problem would correspond to continuation of solution of initial *real* problem (1)?

If there is no precise match of solutions of problems (1) and (2) in the domain D_ε, use of "f.d.m." in finite-difference methods with introduction of common net domain loses its sense. In addition to these questions, the below-given contrary instances will be enough for understanding irrationality of "fictitious domains method".

Contrary instance 1. Let's consider problem of "f.d.m." from the point of view of the second *Newton* law. For this, we will represent equation of vicious fluid dynamics (1) with the help of substantial time derivative

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \mu \Delta \vec{v} + \rho \vec{F}, \tag{5}$$

multiplying which by lementary volume $\delta\tau$ we arrive to formulation of the second *Newton* law

$$\delta m \frac{d\vec{v}}{dt} = \vec{f} \quad , \tag{6}$$

where $\delta m = \rho \delta\tau -$ mass, $\dfrac{d\vec{v}}{dt}$ - acceleration,

$\vec{f} = (-\nabla p + \mu \Delta \vec{v} + \rho \vec{F})\delta\tau$ - resultant (major) force.

Multiplying dynamics equation in system (2) by $\delta\tau$, by analogy with expression (6) we obtain second *Newton* law in the form

$$\delta m \frac{d\vec{v}_\varepsilon}{dt} = \vec{f} - \frac{\vec{v}_\varepsilon}{\varepsilon}\delta\tau \ , \tag{7}$$

where the last member $\frac{\vec{v}_\varepsilon}{\varepsilon}\delta\tau$ determines reactive force, which tents to infinitely large values since $\varepsilon \to 0$. It occurs that by virtue of (7), in fictitious domain the particles of fluid are affected by infinitely large proportional to velocity forces dependent on parameter ε.

Contrary instance 2. Let's start from a simple ordinary differential first-order equation

$$y' = y, \ y(0) = 1, \tag{8}$$

Let's condition that solution of *Cauchy* problem (1) is ought in *real* domain $0 \le x < \infty$. Exact unique solution of the problem is obvious:

$$y(x) = e^x \tag{9}$$

Let domain $-\infty < x \le 0$ according to idea of "f.d.m." be *fictitious* domain. By analogy with (2) this problem (8) is connected to ε-problem

$$y'_\varepsilon = y_\varepsilon - \frac{1}{\varepsilon} y_\varepsilon, y_\varepsilon(0) = 1 \tag{10}$$

Solution of *Cauchy* problem (10) has the following form

$$y_\varepsilon = e^{(1-\frac{1}{\varepsilon})x} \tag{11}$$

Let's note, that initial condition $y_\varepsilon(0) = 1$ in (10) is satisfied by solution (11) with any $\varepsilon \ne 0$. When $\varepsilon \to 0$ solution (11) in point *x=0* will have uncertainty in the index

$$y_\varepsilon(0) = e^{-\frac{0}{\varepsilon \to 0}} \tag{11'}$$

Thus, in “f.d.m.” appears a problem of setting boundary conditions when $\varepsilon \to 0$.

Theorem 1. When $\varepsilon \to 0$ ***solution*** (11) ***of Cauchy fictitious problem (10) may by arbitrary large value differ from extension into fictitious domain of the real solution*** (9) ***of the initial problem (8).***

Proving. Let’s assume $x = -b^2 \in (-\infty, 0], b \neq 0$. For this arbitrary point from fictitious domain, solution (11) of “fictitious” problem will be equal to

$$y_\varepsilon(-b^2) = e^{(1-\frac{1}{\varepsilon})x} = e^x e^{-\frac{x}{\varepsilon}} = e^{-b^2} e^{-\frac{(-b^2)}{\varepsilon}} = e^{-b^2} e^{\frac{b^2}{\varepsilon}} = y(-b^2) e^{\frac{b^2}{\varepsilon}}, \quad (12)$$

where $y(-b^2)$ - extension of real solution (9) in fictitious domain $(-\infty, 0]$. Let’s make a relation

$$\frac{y_\varepsilon(-b^2)}{y(-b^2)} = e^{\frac{b^2}{\varepsilon}}, \quad (13)$$

from which it can be seen, that when $\varepsilon \to 0$ solution (11) $y_\varepsilon(-b^2)$ is different from extention of real solution (9) $y(-b^2)$ into fictitious domain by arbitrary large value, since $e^{\frac{b^2}{\varepsilon}} \to \infty$ tends to infinity, QED.

Let’s consider their difference in the real domain. Let $x = b^2 \in [0, \infty), b \neq 0$. Then, equalities $y_\varepsilon(b^2) = e^{(1-\frac{1}{\varepsilon})x} = e^x e^{-\frac{x}{\varepsilon}} = e^{b^2} e^{-\frac{b^2}{\varepsilon}} = y(b^2) e^{-\frac{b^2}{\varepsilon}}$ result that in the real domain as well difference between solutions (9) and (11) when $\varepsilon \to 0$ also becomes arbitrarily large, because $e^{-\frac{b^2}{\varepsilon}} \to 0$ and fictitious solution tents to zero $y_\varepsilon(b^2) \to 0$, while real solution is nonzero $y(b^2) \neq 0$. In this example, solution (9) of the initial *Cauchy*

problem (8) as well as solution (11) of “f.d.m.” problem (10) clearly satisfies boundary condition $y_\varepsilon(0)=1$, $y(0)=1$.

Contrary instance 3. Let’s consider another example with nonhomogeneous equation

$$y' = y+1,\ y(0)=0, \tag{14}$$

Exact solution of which is the following

$$y(x)=e^x-1$$

(15)

Let’s condition that solution of *Cauchy* problem (14) should be sought in the *real* domain $0\le x<\infty$. Let domain $-\infty<x\le 0$ according to “f.d.m.” be a *fictitious* domain. By analogy with (2) this problem (14) corresponds to ε-problem

$$y'_\varepsilon = y_\varepsilon - \frac{1}{\varepsilon} y_\varepsilon + 1, y_\varepsilon(0)=0 \tag{16}$$

Solution of *Cauchy* problem (16) has the following form

$$y_\varepsilon = e^{(1-\frac{1}{\varepsilon})x} - 1/(1-\frac{1}{\varepsilon}) \tag{17}$$

Solution of fictitious problem (17) approximately satisfies initial condition for $\varepsilon \ne 0$

$$y_\varepsilon(0)=1-1/(1-\frac{1}{\varepsilon}) \tag{18}$$

When $\varepsilon \to 0$ just as in (11’), there arises uncertainty in index

$$y_\varepsilon(0) = e^{-\frac{0}{\varepsilon\to 0}} - 1/(1-\frac{1}{\varepsilon\to 0}) \tag{18’}$$

Let’s assume $x=-b^2\in(-\infty,0]$. Solution (17) in this point will be

$$y_\varepsilon(-b^2) = e^{-b^2} e^{\frac{b^2}{\varepsilon}} - 1/(1-\frac{1}{\varepsilon}) = (e^{-b^2} - 1)e^{\frac{b^2}{\varepsilon}} + e^{\frac{b^2}{\varepsilon}} \text{-}1/(1-\frac{1}{\varepsilon}) =$$

$$= y(-b^2)e^{\frac{b^2}{\varepsilon}} + e^{\frac{b^2}{\varepsilon}} \text{-}1/(1-\frac{1}{\varepsilon}), \tag{19}$$

here $y(-b^2)$ there is an extension into the fictitious domain of the real solution (15). From (19) is seen the infinitely large difference between the fictitious solution $y_\varepsilon(-b^2)$ and actual solution (15) $y(-b^2)$, arising when $\varepsilon \to 0$.

In contrary instances 2 and 3, it is shown that solutions "f.d.m." when $\varepsilon \to 0$ may differ by infinitely large value from extension into fictitious domain of the real solution because of (13) and (19), and these particular solutions in fictitious domain are used in mesh methods as additional conditions for initial problem in the real domain.

Contrary instance 4. Let's show inconsistency of "f.d.m." for flow of viscous incompressible fluid in a channel with parallel walls (*Poiseuille* flow), that is for boundary-value problem. For the given laminal flow, *Navier-Stokes* equations will take the following form /2/:

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} \tag{20}$$

Exact solution of this equation:

$$u = -\frac{1}{2\mu} \cdot \frac{dp}{dx}(b^2 - y^2), \; \frac{dp}{dx} = const \langle 0 \quad , \tag{21}$$

$$u(b) = u(-b) = 0 \tag{22}$$

In fictitious domain D_ε by "f.d.m." ε - equation is solved

$$\frac{dp}{dx} = \mu \frac{d^2u_\varepsilon}{dy^2} - \frac{1}{\varepsilon} u_\varepsilon, \tag{23}$$

Exact solution of this equation in physical domain D with boundary conditions (22) will be:

$$u_{\varepsilon} = -\varepsilon \cdot \frac{dp}{dx}\{1 - \frac{e^{(\frac{1}{\varepsilon\mu})^{\frac{1}{2}} \cdot y} + e^{-y(\frac{1}{\varepsilon\mu})^{\frac{1}{2}}}}{e^{b(\frac{1}{\varepsilon\mu})^{\frac{1}{2}}} + e^{-b(\frac{1}{\varepsilon\mu})^{\frac{1}{2}}}}\} \qquad (24)$$

Difference of solutions (22) and (24)

$$\left|u(y) - u_{\varepsilon}(y)\right| = -\frac{dp}{dx}\left|\frac{1}{2\mu}(b^2 - y^2) + \varepsilon(1 - \frac{e^{y(\frac{1}{\varepsilon\mu})^{\frac{1}{2}}} + e^{-y(\frac{1}{\varepsilon\mu})^{\frac{1}{2}}}}{e^{b(\frac{1}{\varepsilon\mu})^{\frac{1}{2}}} + e^{-b(\frac{1}{\varepsilon\mu})^{\frac{1}{2}}}})\right| \qquad (25)$$

shows that $\lim\limits_{\varepsilon \to 0}\left|u(y) - u_{\varepsilon}(y)\right| \neq 0,$ i.e. for $\varepsilon \to 0$ $u_{\varepsilon}(y)$ does not tend to $u(y)$.

Now, let's consider equation (23) in fictitious domain D_{ε}.

According to "f.d.m." (2) at the boundary S_{ε} of the fictitious domain D_{ε} introduced is homogeneous boundary condition $u_{\varepsilon}(-y_H) = 0$, besides, on the boundary of physical domain given is $u_{\varepsilon}(-b) = 0.$ Under these boundary conditions, exact solution of equation (23) in fictitious domain D_{ε} have the following form:

$$u_{\varepsilon} = -\varepsilon \cdot \frac{dp}{dx}\left[1 - \frac{e^{(\frac{1}{\varepsilon\mu})^{\frac{1}{2}} \cdot (y + \frac{b}{2} + \frac{y_H}{2})} + e^{-(y + \frac{b}{2} + \frac{y_H}{2}) \cdot (\frac{1}{\varepsilon\mu})^{\frac{1}{2}}}}{e^{(\frac{y_H}{2} - \frac{b}{2}) \cdot (\frac{1}{\varepsilon\mu})^{\frac{1}{2}}} + e^{-(\frac{y_H}{2} - \frac{b}{2}) \cdot (\frac{1}{\varepsilon\mu})^{\frac{1}{2}}}}\right] \qquad (26)$$

Similarly to (25) difference of solutions (21) and (26) in fictitious domain D_{ε} does not tent to zero when $\varepsilon \to 0$. This implyies, that

solution (26) in fictitious domain does not add up to solving *Navier-Stokes* equation (21), therefore, in non-physical fictitious domain, D_ε, $\varepsilon -$ quation may not be applied.

Obviously, exact solution of *Navier-Stokes* equations (21) on the boundary "$-y_H$" $\in S_\varepsilon$ of the fictitious domain is nonzero

$$u(-y_H) = -\frac{1}{2\mu} \cdot \frac{dp}{dx}(b^2 - y_H^2) \neq 0\,, \tag{27}$$

therefore, homogeneous boundary condition $u_\varepsilon(-y_H) = 0$ "f.d.m." (2) is fully in contradiction with exact solution (21), according to which $u(-y_H) \neq 0$, which can be seen from Fig. So, in this contrary instance, solution (26) in fictitious domain D_ε by infinitely large value differs from extension of solution (21) into this domain, accordingly, the given inequality of boundary values

$$u_\varepsilon(-y_H) = 0, \quad u(-y_H) \neq 0 \tag{28}$$

proves **senselessness of applying "f.d.m." (2) in problems of hydrodynamics.**

This hydrodynamics contrary instance 3 convincingly shows the problem of setting adequate to extension into fictitious (trans-boundary) domain to solution (27) respective boundary conditions for fictitious problem (23), because homogeneous boundary condition $u_\varepsilon(-y_H) = 0$ "f.d.m." (2) turned out to be fallacious. In this example a simple domain was considered, at this analytical solution (21) of the real problem is known in advance, thanks to which exact bounadary condition (27) is also known.

In solving three-dimensional problems, where analytical solution of type (21), is not known in advance, complexity, frankly speaking, *insolubility* of the problem of setting boundary conditions on fictitious boundary S_ε, not matching the real boundary S, in three-dimentional problems of hydrodynamics, is indisputably obvious.

Given above contrary instances prove that true is

Theorem 2. ***When*** $\varepsilon \to 0$ ***solution of fictitious problem in fictitious domain will not match with extension of solution into fictitious domain of a real problem.***

Main purpose of "f.d.m." is in the use of solution of fictitious problem in fictitious domain as additional boundary conditions in mesh solution techniques for initial primal problem. Contrary instances and the theorem prove fallacy and uselessness of "f.d.m." for solving hydrodynamics equations in nonstandard domains.

Theorem 3. When $\varepsilon \to \infty$ **solution of fictitious problem**

$$\rho[\frac{\partial \vec{v}_\varepsilon}{\partial t} + (\vec{v}_\varepsilon, \nabla)\vec{v}_\varepsilon] + \nabla p_\varepsilon = \mu \Delta \vec{v}_\varepsilon + \rho \vec{F}_\varepsilon - \frac{\vec{v}_\varepsilon - \vec{\varphi}_\varepsilon}{\varepsilon}, (\nabla, \vec{v}_\varepsilon) = 0, \vec{v}_\varepsilon |_{t=0} = \vec{d}_\varepsilon, \vec{v}_\varepsilon |_S = \vec{\varphi}_\varepsilon$$

in fictitious domain will not match with extension of solution into fictitious domain of a real problem:

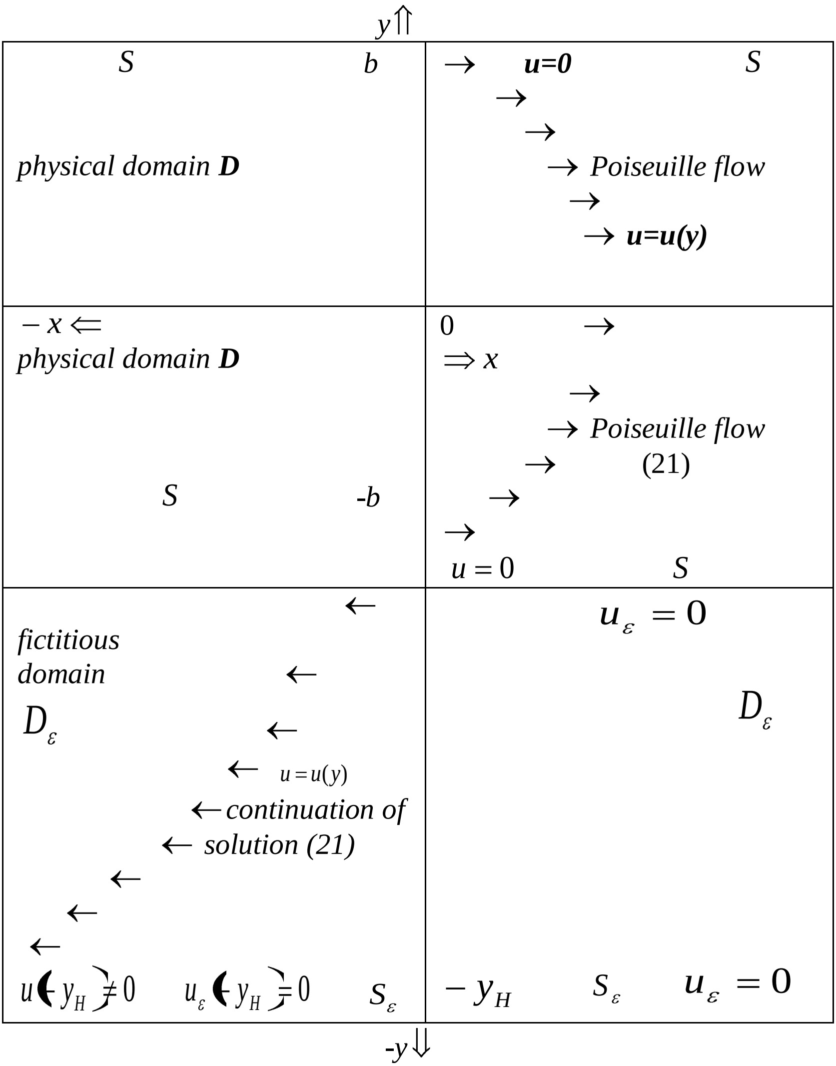
$$\rho[\frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v}] + \nabla p = \mu \Delta \vec{v} + \rho \vec{F}, (\nabla, \vec{v}) = 0, \vec{v} |_{t=0} = \vec{d}, \vec{v} |_S = \vec{\varphi}$$

Theorem proving is supported by previous results and common sense, which is in *absolute impossibility* to set such initial and boundary conditions

$$\vec{v}_\varepsilon |_{t=0} = \vec{d}_\varepsilon, \vec{v}_\varepsilon |_S = \vec{\varphi}_\varepsilon$$

in a fictitious problem, under which solution of a fictitious problem would exactlymatch extension of the initial real problem into fictitious domain (see (28)). It is clear that if there is no such exact fit, solution of a fictitious problem, engaged as an additional condition, will injure solution of a real problem, which is unacceptable.

Suggestion. In problems of flows in domains with solid curved boundary more suitable is extrapolation to mesh points covered by fictitious domain, values of obtained in points of real domain, spline functions formulae can be used for extrapolation, and pressure must be computed from continuity equation. It is much easier and more efficient than solution of a lengthy, not corresponding to laws of physics system of sophisticated equations with ungrounded initially-boundary conditions, in fictitious domain.



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Chapter 6. CONSTRUCTIONS OF MONOTONOUS SIMILAR SCHEMES

§1. Monotonous schemes constructing technology

For building monotonous similar schemes second and more upper order approximation of younger terms effective is next technology (see *Jakupov K.B.* /1/).

Consider initial-boundary task for one-dimensional equation with convection term:

$$\rho\left(\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}\right)=\mu\frac{\partial^2\omega}{\partial x^2}+f,\omega|_{t=0}=d, \qquad (1)$$

$$\omega(0,\mathrm{t})=\varphi_1(t),\omega(a,\mathrm{t})=\varphi_2(t),0\le x\le a,t\in[0,T]$$

Introduce net: spatial

$$\overline{\Omega}_h=\{x_i,i=0,\ldots,N_x;y_j,j=0,1,\ldots,N_y;z_k,k=0,1,\ldots,N_z\}$$

and by time $\overline{\Omega}_\tau=\{t_n=n\tau,n=0,1,\ldots,N_\tau\}$ with steps

$$h_{xi}=x_i-x_{i-1}>0,i=1,2,\ldots,N_x,h_{yj}=y_j-y_{j-1}>0,j=1,2,\ldots,N_y,$$

$$h_{zk}=z_k-z_{k-1}>0,k=1,2,\ldots,N_z,\tau_n=t_n-t_{n-1}>0.$$

Using designations of net functions:

$$f_{ijk}^n\equiv f(x_i,y_j,z_k,t_n),f_{i\pm1jk}^n\equiv f(x_{i\pm1},y_j,z_k,t_n),$$

$$f^n_{ij\pm1k} \equiv f(x_i, y_{j\pm1}, z_k, t_n),\ f^n_{ijk\pm1} \equiv f(x_i, y_j, z_{k\pm1}, t_n),\ f^{n+1}_{ijk} \equiv f(x_i, y_j, z_k, t_{n+1}),$$

Introduce difference derivatives:

$$f^n_x \equiv \frac{f^n_{i+1jk} - f^n_{ijk}}{h_{xi+1}},\ f^n_{\bar{x}} \equiv \frac{f^n_{ijk} - f^n_{i-1jk}}{h_{xi}},\ f^n_{\tilde{x}} \equiv \frac{f^n_{i+1jk} - f^n_{i-1jk}}{h_{xi+1} + h_{xi}}, \hbar_{xi} = \frac{h_{xi+1} + h_{xi}}{2},$$

$$f^n_{\dot{x}} \equiv \frac{f^n_{ijk} - f^n_{i-1jk}}{\hbar_{xi}},\ f^n_{x\dot{x}} \equiv \frac{1}{\hbar_{xi}}\left(\frac{f^n_{i+1jk} - f^n_{ijk}}{h_{xi+1}} - \frac{f^n_{ijk} - f^n_{i-1jk}}{h_{xi}}\right),\ f^n_y \equiv \frac{f^n_{ij+1k} - f^n_{ijk}}{h_{yj+1}},$$

$$f^n_{\bar{y}} \equiv \frac{f^n_{ijk} - f^n_{ij-1k}}{h_{yj}},\ f^n_{\tilde{y}} \equiv \frac{f^n_{ij+1k} - f^n_{ij-1k}}{h_{yj+1} + h_{yj}}, \hbar_{yj} = \frac{h_{yj+1} + h_{yj}}{2},\ f^n_{\dot{y}} \equiv \frac{f^n_{ijk} - f^n_{ij-1k}}{\hbar_{yj}},$$

$$f^n_{y\dot{y}} \equiv \frac{1}{\hbar_{yj}}\left(\frac{f^n_{ij+1k} - f^n_{ijk}}{h_{yj+1}} - \frac{f^n_{ijk} - f^n_{ij-1k}}{h_{yj}}\right),\ f^n_z \equiv \frac{f^n_{ijk+1} - f^n_{ijk}}{h_{zk+1}},$$

$$f^n_{\bar{z}} \equiv \frac{f^n_{ijk} - f^n_{ijk-1}}{h_{zk}},\ f^n_{\tilde{z}} \equiv \frac{f^n_{ijk+1} - f^n_{ijk-1}}{h_{zk+1} + h_{zk}}, \hbar_{zk} = \frac{h_{zk+1} + h_{zk}}{2},$$

$$f^n_{\dot{z}} \equiv \frac{f^n_{ijk} - f^n_{ijk-1}}{\hbar_{zk}},\ f^n_{z\dot{z}} \equiv \frac{1}{\hbar_{zk}}\left(\frac{f^n_{ijk+1} - f^n_{ijk}}{h_{zk+1}} - \frac{f^n_{ijk} - f^n_{ijk-1}}{h_{zk}}\right)$$

Technology /1/ concludes in using decision equation (1) by next way. Take place scheme

$$\frac{\partial\omega}{\partial t} + \frac{|u^n_i| + u^n_i}{2}\left[\frac{\omega^n_i - \omega^n_{i-1}}{h_{xi}} + \frac{h_{xi}}{2}\frac{\partial^2\omega}{\partial x^2} + O(h^2_{xi})\right] + \qquad (2)$$

$$+\frac{u^n_i - |u^n_i|}{2}\left[\frac{\omega^n_{i+1} - \omega^n_i}{h_{xi+1}} - \frac{h_{xi+1}}{2}\frac{\partial^2\omega}{\partial x^2} + O(h^2_{xi})\right] == \bar{\mu}\frac{\partial^2\omega}{\partial x^2} + \bar{f}, \bar{\mu} = \frac{\mu}{\rho}, \bar{f} = \frac{f}{\rho}$$

In (2) is made replacement from (1) $\frac{\partial^2\omega}{\partial x^2} = \frac{1}{\bar{\mu}}\left\{\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x} - \bar{f}\right\}$:

$$\frac{\partial\omega}{\partial t} + \frac{|u^n_i| + u^n_i}{2}\left[\frac{\omega^n_i - \omega^n_{i-1}}{h_{xi}} + \frac{h_{xi}}{2}\frac{1}{\bar{\mu}}\left\{\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x} - \bar{f}\right\} + O(h^2_{xi})\right] + \quad (3)$$

$$+\frac{u_i^n-|u_i^n|}{2}\left[\frac{\omega_{i+1}^n-\omega_i^n}{h_{xi+1}}-\frac{h_{xi+1}}{2}\frac{1}{\bar{\mu}}\left\{\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}-\bar{f}\right\}+O(h_{xi+1}^2)\right]=\bar{\mu}\frac{\partial^2\omega}{\partial x^2}+\bar{f},$$

after that in (3) using approximate

$$\frac{\partial\omega}{\partial t}=\frac{\omega_i^{n+1}-\omega_i^n}{\tau_{n+1}}+O(\tau_{n+1})\equiv\omega_t^n+O(\tau_{n+1}),$$

$$u\frac{\partial\omega}{\partial x}=\frac{|u_i^n|+u_i^n}{2}\left[\frac{\omega_i^n-\omega_{i-1}^n}{h_{xi}}+O(h_{xi})\right]+\frac{u_i^n-|u_i^n|}{2}\left[\frac{\omega_{i+1}^n-\omega_i^n}{h_{xi+1}}+O(h_{xi+1})\right], \quad (4)$$

$$\bar{\mu}\frac{\partial^2\omega}{\partial x^2}=\frac{2\bar{\mu}}{h_{xi+1}+h_{xi}}\left(\frac{\omega_{i+1}^n-\omega_i^n}{h_{xi+1}}-\frac{\omega_i^n-\omega_{i-1}^n}{h_{xi}}\right)+O(\bar{\mu}(h_{xi+1}-h_{xi}))+O(\bar{\mu}\hbar_{xi}^2)$$

Reduction similar terms prove clear scheme 2 order mistake approximation of convection term

$$\rho\left(\omega_t^n+\frac{|u_i^n|+u_i^n}{2}\frac{\omega_i^n-\omega_{i-1}^n}{h_{xi}}+\frac{u_i^n-|u_i^n|}{2}\frac{\omega_{i+1}^n-\omega_i^n}{h_{xi+1}}\right)=$$

$$=\frac{\mu}{1+\frac{\rho}{2\mu}\left(\frac{|u_i^n|+u_i^n}{2}h_{xi}+\frac{|u_i^n|-u_i^n}{2}h_{xi+1}\right)}\left(\frac{\omega_{i+1}^n-\omega_i^n}{h_{xi+1}}-\frac{\omega_i^n-\omega_{i-1}^n}{h_{xi}}\right)\frac{1}{\hbar_{xi}}+f_i^n, \quad (5)$$

In (4) is approximation 1 order. By the same *technology*, use in (3) instead of (4) approximation 2 order

$$u\frac{\partial\omega}{\partial x}=\frac{|u_i^n|+u_i^n}{2}\left[\frac{\omega_i^n-\omega_{i-1}^n}{h_{xi}}+\frac{h_{xi}}{2}\frac{\partial^2\omega}{\partial x^2}+O(h_{xi}^2)\right]+$$

$$+\frac{u_i^n-|u_i^n|}{2}\left[\frac{\omega_{i+1}^n-\omega_i^n}{h_{xi+1}}-\frac{h_{xi+1}}{2}\frac{\partial^2\omega}{\partial x^2}+O(h_{xi+1}^2)\right], \quad (6)$$

further, state $\frac{\partial^2\omega}{\partial x^2}=\frac{1}{\bar{\mu}}\{\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}-\bar{f}\}$ in (6) and put similar term, prove scheme

$$\rho\left(\omega_{t'}^{n}+\frac{|u_i^n|+u_i^n}{2}\frac{\omega_i^n-\omega_{i-1}^n}{h_{xi}}+\frac{u_i^n-|u_i^n|}{2}\frac{\omega_{i+1}^n-\omega_i^n}{h_{xi+1}}\right)= \tag{7}$$

$$=\frac{\mu}{\sum_{k=0}^{M}[\frac{\rho}{2\mu}(\frac{|u_i^n|+u_i^n}{2}h_{xi}+\frac{|u_i^n|-u_i^n}{2}h_{xi+1})]^k}\omega_{x\dot{x}}^n+f_i^n,$$

$$\omega_i^0=d_i, i=1,\ldots,N_x-1, \omega_0^n=\varphi_1^n, \omega_{N_x}^n=\varphi_2^n, \forall n, M=0,1,2,\ldots,$$

which by $\omega_{t'}^n\equiv(\omega_i^{n+1}-\omega_i^n)/\tau$ are clear schemes, convergent and stable in network form *C* by execution of conditions

$$\frac{\rho}{\tau_{n+1}}-\Big[\rho(\frac{|u_i^n|+u_i^n}{2h_{xi}}+\frac{|u_i^n|-u_i^n}{2h_{xi+1}})+$$

$$+\frac{\mu}{\sum_{k=0}^{M}[\frac{\rho}{2\mu}(\frac{|u_i^n|+u_i^n}{2}h_{xi}+\frac{|u_i^n|-u_i^n}{2}h_{xi+1})]^k}\frac{1}{\hbar_{x_i}}(\frac{1}{h_{xi+1}}+\frac{1}{h_{xi}})\Big]\geq 0, \forall(i,n),$$

by $\omega_{t'}^n\equiv(\omega_i^n-\omega_i^{n-1})/\tau$ are implicit schemes, of course convergent and absolutely stable in the same form *C*.

Error of scheme (7) depends on $M\geq 0$. For M = 0 scheme (7) has “approximated ductility” and 1 order of exactness and well-known as *Buleev-Petriscthiv* scheme. For *M* = 1 has 2 order of error of approximation of convection term /1/:

$$\psi_i^n=O(\tau_{n+1})+O(|u_i^n|(h_{xi}^2+h_{xi+1}^2)+O(\mu(h_{xi+1}-h_{xi}))+O(\mu\hbar_{xi}^2),$$

For *M* = 2 on analytical grid $h_{xi}=h_{xi+1}=\hbar_{xi}=h_x, \forall i$, *new scheme (7) takes type*

$$\rho(\omega_t^n+\frac{|u_i^n|+u_i^n}{2}\frac{\omega_i^n-\omega_{i-1}^n}{h_x}+\frac{u_i^n-|u_i^n|}{2}\frac{\omega_{i+1}^n-\omega_i^n}{h_x})=$$

$$=\frac{\mu}{1+\rho\frac{|u_i^n|}{2\mu}h_x+(\rho\frac{|u_i^n|}{2\mu}h_x)^2}(\frac{\omega_{i+1}^n-2\omega_i^n+\omega_{i-1}^n}{h_x^2})+f_i^n$$

For *error of scheme* $z_i^n=\omega_i^n-\overline{\omega}_i^n$, where $\overline{\omega}=\overline{\omega}(x,t)-$ accurate decision of task (1), take place estimation

$$\|z^n\|\le T\max_{0\le k\le n}\|\psi^k\|,n=1,2,\ldots,$$

whence flow convergence of data of monotonous schemes (7), but also stability of schemes by starting condition d and by free term f, that is for disturbance of scheme $\tilde{\omega}^n=\omega^n-\overline{\omega}^n$ take place estimation

$$\|\tilde{\omega}^n\|\le\|\varepsilon\|+T\max_{0\le k\le n}\|\xi^k\|,n=1,2,\ldots.$$

Generalization of this technology on equations in cylindrical, spherical and other coordinate system, but also on multidimentional equations don’t arouse special difficulties.

Necessary notice, that in /3/ for equation with younger derivative

$$(ku')'+r(x)u'-q(x)u=-f(x),$$

monotonous scheme was built by *Samarckyi A.A. technology*

$$\chi(ay_{\bar{x}})_x+b^+a^{(+1)}y_x+b^-ay_{\bar{x}}-dy=-\varphi,\chi=1/(1+R),R=0{,}5|r|h/k,$$

with error *0 (h^2)* on analytical grid.

§2. Monotonous similar schemes 2 order of approximation of convection terms in equations with *constant coefficient* of molecular transport

Above-mentioned technology /1/ for multidimentional equations is applied by similar way.

For two-dimensional equation of parabolic type

$$\rho(\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y})=\mu(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2})+f,\omega|_{t=0}=d,\omega\Big|_S=\varphi,$$

similar (5) clear scheme takes bype

$$\rho\left(\frac{\omega_{ij}^{n+1}-\omega_{ij}^{n}}{\tau_{n+1}}+\frac{|u_{ij}^{n}|+u_{ij}^{n}}{2}\omega_{\bar{x}}^{n}+\frac{u_{ij}^{n}-|u_{ij}^{n}|}{2}\omega_{x}^{n}+\frac{|v_{ij}^{n}|+v_{ij}^{n}}{2}\omega_{\bar{y}}^{n}+\frac{v_{ij}^{n}-|v_{ij}^{n}|}{2}\omega_{y}^{n}\right)=$$

$$=\frac{2\mu_x^v}{h_{xi+1}+h_{xi}}(\omega_x^n-\omega_{\bar{x}}^n)+\frac{2\mu_y^u}{h_{yj+1}+h_{yj}}(\omega_y^n-\omega_{\bar{y}}^n)+f_{ij}^{\,n},$$

where coefficients by dissipative terms are equal

$$Q=1+\frac{\rho}{\mu}\left(\frac{|u_{ij}^{n}|+u_{ij}^{n}}{4}h_{xi}+\frac{|u_{ij}^{n}|-u_{ij}^{n}}{4}h_{xi+1}+\frac{|v_{ij}^{n}|+v_{ij}^{n}}{4}h_{yj}+\frac{|v_{ij}^{n}|-v_{ij}^{n}}{4}h_{yj+1}\right),$$

$$\mu_x^v=\left[\mu+\rho\left(\frac{|v_{ij}^{n}|+v_{ij}^{n}}{4}h_{yj}+\frac{|v_{ij}^{n}|-v_{ij}^{n}}{4}h_{yj+1}\right)\right]/Q,$$

$$\mu_y^u=\left[\mu+\rho\left(\frac{|u_{ij}^{n}|+u_{ij}^{n}}{4}h_{xi}+\frac{|u_{ij}^{n}|-u_{ij}^{n}}{4}h_{xi+1}\right)\right]/Q$$

On analytical grid $h_{yj+1}=h_{yj}=h_y,\forall j, h_{xi+1}=h_{xi}=h_x,\forall i$ these coefficient are simplified

$$\mu_x^v=\left(\mu+\rho\frac{|v_{ij}^{n}|h_y}{2}\right)/\left[1+\rho\left(\frac{|u_{ij}^{n}|h_x}{2\mu}+\frac{|v_{ij}^{n}|h_y}{2\mu}\right)\right],$$

$$\mu_y^u=\left(\mu+\rho\frac{|u_{ij}^{n}|h_x}{2}\right)/\left[1+\rho\left(\frac{|u_{ij}^{n}|h_x}{2\mu}+\frac{|v_{ij}^{n}|h_y}{2\mu}\right)\right]$$

The scheme is stable and converge in form C by execution of conditions

$$\frac{\rho}{\tau_{n+1}}-\left[\rho\left(\frac{|u_{ij}^{n}|+u_{ij}^{n}}{2h_{xi}}+\frac{|u_{ij}^{n}|-u_{ij}^{n}}{2h_{xi+1}}+\frac{|v_{ij}^{n}|+v_{ij}^{n}}{2h_{yj}}+\frac{|v_{ij}^{n}|-v_{ij}^{n}}{2h_{yj+1}}\right)\right.$$

$$\left.+\frac{2\mu_x^v}{h_{xi+1}+h_{xi}}\left(\frac{1}{h_{xi+1}}+\frac{1}{h_{xi}}\right)+\frac{2\mu_y^u}{h_{yj+1}+h_{yj}}\left(\frac{1}{h_{yj+1}}+\frac{1}{h_{yj}}\right)\right]\geq 0,\quad \forall(i,j,n)$$

It is not difficult to build by this methodology absolutely convergent and stable schemes of *Krank – Nicholson* type or schemes of method of fractional steps. Exactly in work /1/ similar approximation is applied for *Helmholtz* equations in scheme of variable directions.

For three dimensional equation of parabolic type

$$\rho(\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}+w\frac{\partial\omega}{\partial z})=\mu(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}+\frac{\partial^2\omega}{\partial z^2})+f,$$

monotonous similar clear scheme has type

$$\rho(\frac{\omega_{ijk}^{n+1}-\omega_{ijk}^{n}}{\tau_{n+1}}+\frac{|u_{ijk}^n|+u_{ijk}^n}{2}\omega_{\bar{x}}^n+\frac{u_{ijk}^n-|u_{ijk}^n|}{2}\omega_{x}^n+\frac{|v_{ijk}^n|+v_{ijk}^n}{2}\omega_{\bar{y}}^n+\frac{v_{ijk}^n-|v_{ijk}^n|}{2}\omega_{y}^n+$$

$$+\frac{|w_{ijk}^n|+w_{ijk}^n}{2}\omega_{\bar{z}}^n+\frac{w_{ijk}^n-|w_{ijk}^n|}{2}\omega_{z}^n)=\frac{2\mu_x^{vw}}{h_{xi+1}+h_{xi}}(\omega_x^n-\omega_{\bar{x}}^n)+\frac{2\mu_y^{wu}}{h_{yj+1}+h_{yj}}(\omega_y^n-\omega_{\bar{y}}^n)+$$

$$+\frac{2\mu_z^{uv}}{h_{zk+1}+h_{zk}}(\omega_z^n-\omega_{\bar{z}}^n)+f_{ijk}^n, i=1,\ldots,N_x-1, j=1,\ldots,N_y-1, k=1,\ldots,N_z-1,$$

where coefficients by dissipative terms are equal

$$Q=1+\frac{\rho}{\mu}(\frac{|u_{ijk}^n|+u_{ijk}^n}{4}h_{xi}+\frac{|u_{ijk}^n|-u_{ijk}^n}{4}h_{xi+1}+\frac{|v_{ijk}^n|+v_{ijk}^n}{4}h_{yj}+\frac{|v_{ijk}^n|-v_{ijk}^n}{4}h_{yj+1}+$$

$$+\frac{|w_{ijk}^n|+w_{ijk}^n}{4}h_{zk}+\frac{|w_{ijk}^n|-w_{ijk}^n}{4}h_{zk+1}),$$

$$\mu_x^{vw}=[\mu+\rho(\frac{|v_{ijk}^n|+v_{ijk}^n}{4}h_{yj}+\frac{|v_{ijk}^n|-v_{ijk}^n}{4}h_{yj+1}+\frac{|w_{ijk}^n|+w_{ijk}^n}{4}h_{zk}+\frac{|w_{ijk}^n|-w_{ijk}^n}{4}h_{zk+1})]/Q,$$

$$\mu_y^{wu}=[\mu+\rho(\frac{|w_{ijk}^n|+w_{ijk}^n}{4}h_{zk}+\frac{|w_{ijk}^n|-w_{ijk}^n}{4}h_{zk+1}+\frac{|u_{ijk}^n|+u_{ijk}^n}{4}h_{xi}+\frac{|u_{ijk}^n|-u_{ijk}^n}{4}h_{xi+1})]/Q,$$

$$\mu_z^{uv}=[\mu+\rho(\frac{|u_{ijk}^n|+u_{ijk}^n}{4}h_{xi}+\frac{|u_{ijk}^n|-u_{ijk}^n}{4}h_{xi+1}+\frac{|v_{ijk}^n|+v_{ijk}^n}{4}h_{yj}+\frac{|v_{ijk}^n|-v_{ijk}^n}{4}h_{yj+1})]/Q$$

On analytical grid $h_{xi+1}=h_{xi}=h_x, h_{yj+1}=h_{yj}=h_y, h_{zk+1}=h_{zk}=h_z$ these coefficients are simplified

$$Q=1+\frac{\rho}{\mu}(\frac{|u_{ijk}^n|}{2}h_x+\frac{|v_{ijk}^n|}{2}h_y+\frac{|w_{ijk}^n|}{2}h_z), \mu_x^{vw}=[\mu+\rho(\frac{|v_{ijk}^n|}{2}h_y+\frac{|w_{ijk}^n|}{2}h_z)]/Q,$$

$$\mu_y^{wu}=[\mu+\rho(\frac{|w_{ijk}^n|}{2}h_z+\frac{|u_{ijk}^n|}{2}h_x)]/Q, \mu_z^{uv}=[\mu+\rho(\frac{|u_{ijk}^n|}{2}h_x+\frac{|v_{ijk}^n|}{2}h_y)]/Q$$

Condition of stable and convergence in form *C*:

$$\frac{\rho}{\tau_{n+1}} - \Big[\rho\Big(\frac{|u^n_{ijk}|+u^n_{ijk}}{2h_{xi}} + \frac{|u^n_{ijk}|-u^n_{ijk}}{2h_{xi+1}} + \frac{|v^n_{ijk}|+v^n_{ijk}}{2h_{yj}} + \frac{|v^n_{ijk}|-v^n_{ijk}}{2h_{yj+1}} +$$

$$+\frac{|w^n_{ijk}|+w^n_{ijk}}{2h_{zk}} + \frac{|w^n_{ijk}|-w^n_{ijk}}{2h_{zk+1}}\Big) + \frac{2\mu^{vw}_x}{h_{xi+1}+h_{xi}}\Big(\frac{1}{h_{xi+1}}+\frac{1}{h_{xi}}\Big)+$$

$$+\frac{2\mu^{wu}_y}{h_{yj+1}+h_{yj}}\Big(\frac{1}{h_{yj+1}}+\frac{1}{h_{yj}}\Big)+\frac{2\mu^{uv}_z}{h_{zk+1}+h_{zk}}\Big(\frac{1}{h_{zk+1}}+\frac{1}{h_{zk}}\Big)\Big] \geq 0,\ \ \forall(i,j,k,n),$$

§3. Monotonous similar schemes 2 order of approximation of convective terms in equations with *variable coefficient* of molecular transport

Consider similar equation with variable coefficient of molecular transport $\mu = \mu(x,t) > 0$:

$$\rho\Big(\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x}\Big) = \frac{\partial}{\partial x}\Big(\mu\frac{\partial\omega}{\partial x}\Big) + f,$$

$$\omega|_{t=0} = d, \omega(0,t) = \varphi_1(t), \omega(a,t) = \varphi_2(t), 0 \le x \le a, t \in [0,T]$$

For differentiable function $\mu = \mu(x,t)$ this equation may be transformed to type

$$\rho\frac{\partial\omega}{\partial t} + \Big(\rho u - \frac{\partial\mu}{\partial x}\Big)\frac{\partial\omega}{\partial x} = \mu\frac{\partial^2\omega}{\partial x^2} + f,$$

and to build for it monotonous scheme **§2**.

In case of disruptive non-differentiable coefficient $\mu = \mu(x,t)$, apparently, this transformation is not apply. There is technology of building of monotonous schemes 2 order of approximation of convection terms to apply to equivalent equation

$$\rho\Big(\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x}\Big) = \frac{\partial}{\partial x}\Big[(\mu - \mu_{\min})\frac{\partial\omega}{\partial x}\Big] + \mu_{\min}\frac{\partial^2\omega}{\partial x^2} + f\ ,$$

where indicated in $\mu_{\min}$ minimal value of function

$\mu(x,t) \geq \mu_{\min} = const$. By technology **§1** proved monotonous schemes of equivalent equation for *k>1*:

$$\rho(\omega_t^n + \frac{|u_i^n| + u_i^n}{2} \frac{\omega_i^n - \omega_{i-1}^n}{h_{xi}} + \frac{u_i^n - |u_i^n|}{2} \frac{\omega_{i+1}^n - \omega_i^n}{h_{xi+1}}) =$$

$$= \left\{ \frac{\mu_{\min}}{\sum_{k=0}^{M} [\frac{\rho}{2\mu_{\min}} (\frac{|u_i^n| + u_i^n}{2} h_{xi} + \frac{|u_i^n| - u_i^n}{2} h_{xi+1})]^k} - \mu_{\min} \right\} *$$

$$* \frac{1}{\hbar_{xi}} (\frac{\omega_{i+1}^n - \omega_i^n}{h_{xi+1}} - \frac{\omega_i^n - \omega_{i-1}^n}{h_{xi}}) +$$

$$+ \frac{1}{\hbar_{xi}} (\frac{\mu_{i+1}^n + \mu_i^n}{2} \frac{\omega_{i+1}^n - \omega_i^n}{h_{xi+1}} - \frac{\mu_i^n + \mu_{i-1}^n}{2} \frac{\omega_i^n - \omega_{i-1}^n}{h_{xi}}) + f_i^n,$$

with approximation 2 order and above of convective term.

For two-dimensional equation of parabolic type

$$\rho(\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}) = \frac{\partial}{\partial x}(\mu \frac{\partial \omega}{\partial x}) + \frac{\partial}{\partial y}(\mu \frac{\partial \omega}{\partial y}) + f, \omega|_{t=0} = d, \omega\Big|_S = \varphi$$

monotonous schemes are similar by construction:

$$\rho(\frac{\omega_{ij}^{n+1} - \omega_{ij}^n}{\tau_{n+1}} + \frac{|u_{ij}^n| + u_{ij}^n}{2} \omega_{\bar{x}}^n + \frac{u_{ij}^n - |u_{ij}^n|}{2} \omega_x^n + \frac{|v_{ij}^n| + v_{ij}^n}{2} \omega_{\bar{y}}^n + \frac{v_{ij}^n - |v_{ij}^n|}{2} \omega_y^n) =$$

$$= (\mu_{uxv}^v - \mu_{\min}) \frac{1}{\hbar_{xi}} (\omega_x^n - \omega_{\bar{x}}^n) + (\mu_{vyu}^u - \mu_{\min}) \frac{1}{\hbar_{yj}} (\omega_y^n - \omega_{\bar{y}}^n) +$$

$$+ \frac{1}{\hbar_{xi}} (\frac{\mu_{i+1j}^n + \mu_{ij}^n}{2} \omega_x^n - \frac{\mu_{ij}^n + \mu_{i-1j}^n}{2} \omega_{\bar{x}}^n) +$$

$$+ \frac{1}{\hbar_{yj}} (\frac{\mu_{ij+1}^n + \mu_{ij}^n}{2} \omega_y^n - \frac{\mu_{ij}^n + \mu_{ij-1}^n}{2} \omega_{\bar{y}}^n) + f_{ij}^n, i = 1,\ldots,N_x - 1, j = 1,\ldots,N_y - 1;$$

where coefficients by dissipative terms are equal

$$Q=1+\frac{\rho}{\mu_{\min}}(\frac{|u_{ij}^{n}|+u_{ij}^{n}}{4}h_{xi}+\frac{|u_{ij}^{n}|-u_{ij}^{n}}{4}h_{xi+1}+\frac{|v_{ij}^{n}|+v_{ij}^{n}}{4}h_{yj}+\frac{|v_{ij}^{n}|-v_{ij}^{n}}{4}h_{yj+1}),$$

$$\mu_{x}^{v}=[\mu_{\min}+\rho(\frac{|v_{ij}^{n}|+v_{ij}^{n}}{4}h_{yj}+\frac{|v_{ij}^{n}|-v_{ij}^{n}}{4}h_{yj+1})]/Q,$$

$$\mu_{y}^{u}=[\mu_{\min}+\rho(\frac{|u_{ij}^{n}|+u_{ij}^{n}}{4}h_{xi}+\frac{|u_{ij}^{n}|-u_{ij}^{n}}{4}h_{xi+1})]/Q$$

For three-dimensional equation of parabolic type

$$\rho(\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}+w\frac{\partial\omega}{\partial z})=\frac{\partial}{\partial x}(\mu\frac{\partial\omega}{\partial x})+\frac{\partial}{\partial y}(\mu\frac{\partial\omega}{\partial y})+\frac{\partial}{\partial z}(\mu\frac{\partial\omega}{\partial z})+f,$$

monotonous schemes have type:

$$\rho(\frac{\omega_{ijk}^{n+1}-\omega_{ijk}^{n}}{\tau_{n+1}}+\frac{|u_{ijk}^{n}|+u_{ijk}^{n}}{2}\omega_{\bar{x}}^{n}+\frac{u_{ijk}^{n}-|u_{ijk}^{n}|}{2}\omega_{x}^{n}+\frac{|v_{ijk}^{n}|+v_{ijk}^{n}}{2}\omega_{\bar{y}}^{n}+\frac{v_{ijk}^{n}-|v_{ijk}^{n}|}{2}\omega_{y}^{n}+$$

$$+\frac{|w_{ijk}^{n}|+w_{ijk}^{n}}{2}\omega_{\bar{z}}^{n}+\frac{w_{ijk}^{n}-|w_{ijk}^{n}|}{2}\omega_{z}^{n})=(\mu_{x}^{vw}-\mu_{\min})\frac{1}{\hbar_{xi}}(\omega_{x}^{n}-\omega_{\bar{x}}^{n})+$$

$$+(\mu_{y}^{wu}-\mu_{\min})\frac{1}{\hbar_{yj}}(\omega_{y}^{n}-\omega_{\bar{y}}^{n})+(\mu_{z}^{uv}-\mu_{\min})\frac{1}{\hbar_{zk}}(\omega_{z}^{n}-\omega_{\bar{z}}^{n})+$$

$$+\frac{1}{\hbar_{xi}}(\frac{\mu_{i+1jk}^{n}+\mu_{ijk}^{n}}{2}\omega_{x}^{n}-\frac{\mu_{ijk}^{n}+\mu_{i-1jk}^{n}}{2}\omega_{\bar{x}}^{n})+\frac{1}{\hbar_{xi}}(\frac{\mu_{ij+1k}^{n}+\mu_{ijk}^{n}}{2}\omega_{y}^{n}-\frac{\mu_{ijk}^{n}+\mu_{ij-1k}^{n}}{2}\omega_{\bar{y}}^{n})+$$

$$+\frac{1}{\hbar_{xi}}(\frac{\mu_{ijk+1}^{n}+\mu_{ijk}^{n}}{2}\omega_{z}^{n}-\frac{\mu_{ijk}^{n}+\mu_{ijk-1}^{n}}{2}\omega_{\bar{z}}^{n})+f_{ijk}^{n},x\in\Omega_{h},$$

coefficients by dissipative terms are equal

$$Q=1+\frac{\rho}{\mu_{\min}}(\frac{|u_{ijk}^{n}|+u_{ijk}^{n}}{4}h_{xi}+\frac{|u_{ijk}^{n}|-u_{ijk}^{n}}{4}h_{xi+1}+\frac{|v_{ijk}^{n}|+v_{ijk}^{n}}{4}h_{yj}+\frac{|v_{ijk}^{n}|-v_{ijk}^{n}}{4}h_{yj+1}+$$

$$+\frac{|w_{ijk}^{n}|+w_{ijk}^{n}}{4}h_{zk}+\frac{|w_{ijk}^{n}|-w_{ijk}^{n}}{4}h_{zk+1}),$$

$$\mu_x^{vw} = [\mu_{\min} + \rho(\frac{|v_{ijk}^n| + v_{ijk}^n}{4} h_{yj} + \frac{|v_{ijk}^n| - v_{ijk}^n}{4} h_{yj+1} + \frac{|w_{ijk}^n| + w_{ijk}^n}{4} h_{zk} + \frac{|w_{ijk}^n| - w_{ijk}^n}{4} h_{zk+1})]/Q,$$

$$\mu_y^{wu} = [\mu_{\min} + \rho(\frac{|w_{ijk}^n| + w_{ijk}^n}{4} h_{zk} + \frac{|w_{ijk}^n| - w_{ijk}^n}{4} h_{zk+1} + \frac{|u_{ijk}^n| + u_{ijk}^n}{4} h_{xi} + \frac{|u_{ijk}^n| - u_{ijk}^n}{4} h_{xi+1})]/Q,$$

$$\mu_z^{uv} = [\mu_{\min} + \rho(\frac{|u_{ijk}^n| + u_{ijk}^n}{4} h_{xi} + \frac{|u_{ijk}^n| - u_{ijk}^n}{4} h_{xi+1} + \frac{|v_{ijk}^n| + v_{ijk}^n}{4} h_{yj} + \frac{|v_{ijk}^n| - v_{ijk}^n}{4} h_{yj+1})]/Q$$

Condition of stability of clear scheme

$$\frac{\rho}{\tau_{n+1}} - \Big[\rho(\frac{|u_{ijk}^n| + u_{ijk}^n}{2h_{xi}} + \frac{|u_{ijk}^n| - u_{ijk}^n}{2h_{xi+1}} + \frac{|v_{ijk}^n| + v_{ijk}^n}{2h_{yj}} + \frac{|v_{ijk}^n| - v_{ijk}^n}{2h_{yj+1}} +$$

$$+ \frac{|w_{ijk}^n| + w_{ijk}^n}{2h_{zk}} + \frac{|w_{ijk}^n| - w_{ijk}^n}{2h_{zk+1}}) + (\mu_x^{vw} - \mu_{\min}) \frac{1}{\hbar_{xi}} (\frac{1}{h_{xi+1}} + \frac{1}{h_{xi}}) +$$

$$+ (\mu_y^{wu} - \mu_{\min}) \frac{1}{\hbar_{yj}} (\frac{1}{h_{yj+1}} + \frac{1}{h_{yj}}) + (\mu_y^{uv} - \mu_{\min}) \frac{1}{\hbar_{zk}} (\frac{1}{h_{zk+1}} + \frac{1}{h_{zk}}) +$$

$$+ \frac{1}{\hbar_{xi}} (\frac{\mu_{i+1jk}^n + \mu_{ijk}^n}{2h_{xi+1}} + \frac{\mu_{ijk}^n + \mu_{i-1jk}^n}{2h_{xi}}) + \frac{1}{\hbar_{yj}} (\frac{\mu_{ij+1k}^n + \mu_{ijk}^n}{2h_{yj+1}} + \frac{\mu_{ijk}^n + \mu_{ij-1k}^n}{2h_{yj}}) +$$

$$+ \frac{1}{\hbar_{zk}} (\frac{\mu_{ijk+1}^n + \mu_{ijk}^n}{2h_{zk+1}} + \frac{\mu_{ijk}^n + \mu_{ijk-1}^n}{2h_{zk}}) \Big] \geq 0, \quad \forall (i, j, k, n)$$

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Chapter 7. PARADOXES OF DEFORMATIONS THEORY AND ELASTICITY THEORY EQUATIONS OF NAVIER-CAUCHY-LAME

§1. Paradoxes of deformations theory

On page 64 in «Mechanics of continua, V 1» by *Sedov L.I.*, in paragraph **«On dependency of attendant system basis vector on time» it is stated: «…**Indeed, in motion of deformed body, distances between its points *M* and *M'* change. Coordinate curves of attendant coordinates system are deformed, ***and* basis** $\vec{\ni}_i$ ***vectors change in <u>time</u> so that their values and angles between them change as well*…».** Let's remind that in the «Basic course of theoretical mechanics, part 1» of *Buchgolz N.N.* for varying vector $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$, with basis changing in time $\vec{i}, \vec{j}, \vec{k}$ in moving body-fixed attendant coordinates *Oxyz,* time derivative is defined as:

$$\frac{d\vec{a}}{dt} = \frac{da_x}{dt}\vec{i} + \frac{da_y}{dt}\vec{j} + \frac{da_z}{dt}\vec{k} + a_x\frac{d\vec{i}}{dt} + a_y\frac{d\vec{j}}{dt} + a_z\frac{d\vec{k}}{dt} \quad ,$$

further by *Euler* formula $\frac{d\vec{i}}{dt} = [\vec{\omega}, \vec{i}\,], \frac{d\vec{j}}{dt} = [\vec{\omega}, \vec{j}\,], \frac{d\vec{k}}{dt} = [\vec{\omega}, \vec{k}\,],$

where $\vec{\omega}$ - angular rotational velocity.

Reducing by *dt,* we have

$$d\vec{a} = da_x\vec{i} + da_y\vec{j} + da_z\vec{k} + a_x d\vec{i} + a_y d\vec{j} + a_z d\vec{k} \qquad (1)$$

Therefore, on page 65 in the same section **«On dependency of attendant system basis vector on time»** of the cited book by *Sedov* /1/ expressions

$$d\vec{r} = d\xi^i\vec{\ni}_i \text{ and } d\vec{r}\,' = d\xi^i\vec{\ni}_i' \qquad (2)$$

conflict with formula (1), and, accordingly, must be replaced with corresponding to formula (1) expressions

$$d\vec{r} = d\xi^i\vec{\ni}_i + \xi^i d\vec{\ni}_i\,,\ d\vec{r}\,' = d\xi^i\vec{\ni}_i' + \xi^i d\vec{\ni}_i' \qquad (3)$$

with all resulting consequences in the deformations theory. Strictly speaking, doubtfulness of formula (2) from the point of view of formula (1) is confirmed by statement that «… ***and basis*** $\vec{\ni}_i$ ***vectors***

are changing in <u>time</u> so, that their values and angles between them change…» because $\vec{Э}'_i$, i=1,2,3 – basis vectors at the initial moment of time t_0, $\hat{Э}_i$, i=1,2,3 – basis vectors at the current moment of time t, *and also* $\vec{r}$ and $\vec{r}\,'$ and, as already said, variables. According to *Sedov* /1/ for representations (2) it is assumed

$$|d\vec{r}\,| = ds = \sqrt{\hat{g}_{ij} d\xi^i d\xi^j}\,, \hat{g}_{ij} = \hat{Э}_i \cdot \hat{Э}_j,$$

$$|d\vec{r}\,'| = ds' = \sqrt{g'_{ij} d\xi^i d\xi^j}\,, g'_{ij} = \vec{Э}'_i \cdot \vec{Э}'_j \qquad (4)$$

obviously, for correct representations (3) instead of (4) obtained are absolutely other expressions, since

$$|d\vec{r}\,| = (d\xi^i \hat{Э}_i + \xi^i d\hat{Э}_i, d\xi^i \hat{Э}_i + \xi^i d\hat{Э})^{\frac{1}{2}},$$

$$|d\vec{r}\,'| = (d\xi^i \vec{Э}'_i + \xi^i d\vec{Э}'_i, d\xi^i \vec{Э}'_i + \xi^i d\vec{Э}'_i)^{\frac{1}{2}} \qquad (5)$$

Further on, based on (4) derived is formula /1/:

$$\varepsilon_{ij} = \frac{1}{2}(\dot{\nabla}_i \dot{w}_j + \dot{\nabla}_j \dot{w}_i - \dot{\nabla}_i \dot{w}_k \dot{\nabla}_j \dot{w}_k) \qquad (6)$$

Product of derivatives $\dot{\nabla}_i \dot{w}_k \dot{\nabla}_j \dot{w}^k$ in (6) *is commonly supposed to be a small value* and it is neglected, thus (6) is simplified down to the following known expression of motion tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial w_j}{\partial x_i} + \frac{\partial w_i}{\partial x_j}\right), \quad i, j = 1,2,3\,, \qquad (7)$$

If $\dot{\nabla}_i \dot{w}_k \dot{\nabla}_j \dot{w}^k$ is not small, (7) does not take place. Obviously, due to (5) instead of (6) and (7) there will be absolutely *different relations*. (From the point of view of common sense, formula (6) if absurd, *if* w_j and x_j *would* have different dimensionalities, for example, [w_j] =m/c, and [x_j]=m). The same circumstance is present in the book by *G. Mase* «Theory and problems of continua mechanics», if to bring *Mase*'s symbols into compliance with symbols of *Sedov*: *d***x**= $d\vec{r}$ **,** *d***X**= $d\vec{r}\,'$ **,** $\xi^1 = x_1, \xi^2 = x_2$, $\xi^3 = x_3$, $\varepsilon_{ij} = E_{ij}, w_i = u_i$**,** ***w*=*u*** etc. Formula (6) in /2/ has the following form

$$E_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}) \tag{8}$$

Thus, remarks (2) and (3), (5) are related to derivation of formula (8). It is considered that in (8) gradients are small in comparison with one and they can be neglected. Strictly speaking, and it is evident, that dropping of gradients $\dot{\nabla}_i \dot{w}_k \dot{\nabla}_j \dot{w}^k$ and $\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}$ is connected with *slanting* (7) to *false symmetry of continuum strains tensor*. Continuum strains tensor in the general case is not symmetric (refer to **Chapter 1**).

§2. Alternative presentation of relative movement *d*u

In symbols of *G.Mase* /2/ relative movement vector *d***u** (in the book by *Sedov L.I.*/1/ - denoted as *d**w***) is introduced as difference of vectos of oint Q_0 initial position and final position of the same point P_0: *d***u=u**$^{(Q_0)}$ **-u**$^{(P_0)}$**.** Expansion of **u**$^{(P_0)}$ in *Taylor* series into neighborhood of point P_0 gives *d***u=К***d***x,** or in componentwise representation (inexact differential)

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j \text{ , } i=1,2,3 \tag{1}$$

(by index *j* summing up from 1 to 3 is performed).

Mark that tensor **К is nonsymmetric:**

$$\mathbf{K}= \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \tag{2}$$

In /1/-/3/ expansion (1) is traditionally reduced to the form

$$du_i = [\frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i})]dx_j\,, \qquad (3)$$

or in symbols of *Sedov* /1/

$$dw_i = [\frac{1}{2}(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}) + \frac{1}{2}(\frac{\partial w_i}{\partial x_j} - \frac{\partial w_j}{\partial x_i})]dx_j$$

First sum in (3) is represented by *Euler tensor of linear deformation*

$$\mathbf{E} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}) \\ \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}) & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \qquad (4)$$

This is a symmetric matrix; second part in (3) are *components of Euler linear rotation tensor*

$$\omega_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}) \qquad (5)$$

At the same time they are components of antisymmetric tensor

$$\mathbf{R} = \begin{pmatrix} 0 & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}) & 0 & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}) & 0 \end{pmatrix}$$

Obviously *Taylor* series is presented in two ways in the form (1) and (3):

$$d\mathbf{u}=\mathbf{K}d\mathbf{x} \quad \text{и} \quad d\mathbf{u}=\mathbf{E}d\mathbf{x}+\frac{1}{2}[\mathrm{rot}\mathbf{u},d\mathbf{x}] \qquad (6)$$

or with the help of antisymmetric matrix as follows

$$d\mathbf{u}=\mathbf{E}d\mathbf{x}+\mathbf{R}d\mathbf{x}] \qquad (7)$$

In addition to (6), (7) there is the infinite number of various forms of representing *Taylor* series (1). With this purpose we'll introduce a family of one-parametric matrices

$$\mathbf{C}_b=\begin{pmatrix} \frac{\partial u_1}{\partial x_1}, & \left(\frac{b-1}{b}\frac{\partial u_2}{\partial x_1}+\frac{1}{b}\frac{\partial u_1}{\partial x_2}\right), & \left(\frac{b-1}{b}\frac{\partial u_3}{\partial x_1}+\frac{1}{b}\frac{\partial u_1}{\partial x_3}\right) \\ \left(\frac{b-1}{b}\frac{\partial u_1}{\partial x_2}+\frac{1}{b}\frac{\partial u_2}{\partial x_1}\right), & \frac{\partial u_2}{\partial x_2}, & \left(\frac{b-1}{b}\frac{\partial u_3}{\partial x_2}+\frac{1}{b}\frac{\partial u_2}{\partial x_3}\right) \\ \left(\frac{b-1}{b}\frac{\partial u_1}{\partial x_3}+\frac{1}{b}\frac{\partial u_3}{\partial x_1}\right), & \left(\frac{b-1}{b}\frac{\partial u_2}{\partial x_3}+\frac{1}{b}\frac{\partial u_3}{\partial x_2}\right), & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \qquad (8)$$

Taylor series (1) with the use of matrices (8) has the infinite number of alternative or universal representations

$$d\mathbf{u}=\mathbf{C}_b\, d\mathbf{x}+\frac{b-1}{b}[\mathrm{rot}\mathbf{u},d\mathbf{x}], \quad b\neq 0, |b|<\infty \qquad (9)$$

When $b=1$, *it comes out that* $\mathbf{C}_1=\mathbf{K}$, when $b=2$ - $\mathbf{C}_2=\mathbf{E}$ etc.

§3. Paradoxes of *Navier-Cauchy-Lame* hypothesis

For applying *Hooke* law in the theory of deformable solids *hypothesis* of *Navier-Cauchy-Lame /2/* is used, stating that in *Taylor* series (or inexact differential)

$$du_i=[\frac{1}{2}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})+\frac{1}{2}(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i})]dx_j \qquad (1)$$

for defining components of strain tensor first half of this series is enough

$$du_i = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})dx_j, \qquad (2)$$

second part of the series is neglected, assuming that

$$\frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}) = 0, \qquad (3)$$

although both these expressions consist of the same gradients.

As a result, according to *Navier-Cauchy-Lame* hypothesis, *Hooke* law was defined and is widely used in the following form

$$\pi_{ji} = \lambda\delta_{ij} div\vec{u} + 2\mu\varepsilon_{ji}, \varepsilon_{ji} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), i, j = 1,2,3, \qquad (4)$$

where $\delta_{ij} = \begin{cases} 0, i \neq j, \\ 1, i = j, \end{cases}$, $\lambda, \mu - Lame$ coefficients, $\pi_{ij} = \pi_{ji}$.

Thus, according to (4) it is stated that forces deforming a body create only *linear deformation* E*d*x. According to *Navier-Cauchy-Lame* hypothesis *Euler linear rotation* is equated to zero $\frac{1}{2}$[rot**u,***d***x]=R***d***x=0,** which is specifically expressed because of (3) as equality of displacement curl to zero (see Lurie /3/):

$$\mathrm{rot}\mathbf{u}=\mathbf{0} \qquad (5)$$

(in particular, solvability condition of *A.N.Konovalov* has the form /4/ $\int_D rot\vec{u}dM = 0$)

Accordingly, formula (1) is cut and takes a shortened form (2), far from *Taylor* series:

$$d\mathbf{u}=\mathbf{E}d\mathbf{x} \qquad (6)$$

Of course, the indicated paradox is connected with slanting (4) to formula of *Euler* tensor of finite deformations

$$E_{ji} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}),$$

in which, according to small deformations theory products of $\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}$, that can be quite large are dropped, and suggested is doubtful formula $E_{ji} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$.

So, if to follow *Navier-Cauchy-Lame* formula, according to which rot**u=0,** than from representation of *Taylor* series in universal form

$$d\mathbf{u} = \mathbf{C}_b\, d\mathbf{x} + \frac{b-1}{b}[\mathrm{rot}\mathbf{u}, d\mathbf{x}], \quad b \neq 0, |b| < \infty \qquad (7)$$

dropped is the second addend and, based on expression

$$d\mathbf{u} = \mathbf{C}_b\, d\mathbf{x}, \qquad (8)$$

logically, *Hooke* law must be defined in the form

$$\pi_{ji} = \lambda\delta_{ij} div\vec{u} + b\mu\varepsilon_{ji}, \varepsilon_{ji} = \frac{1}{b}\frac{\partial u_i}{\partial x_j} + \frac{b-1}{b}\frac{\partial u_j}{\partial x_i},\ b \neq 0, |b| < \infty \qquad (9)$$

In formulae (4) *Navier-Cauchy-Lame* strains tensor is symmetric, in formulae (7) there are infinite many nonsymmetric strain tensors when $b \neq 2$. Formula (7) is the Taylor series with any values of $b \neq 0$, including for $b = 1$. Hooke law (9) was formulated for representation (8), obtained from (7) when rot**u=0, and for** $b = 1$ takes the following form

$$\pi_{ji} = \lambda\delta_{ij} div\vec{u} + \mu\varepsilon_{ji}, \varepsilon_{ji} = \frac{\partial u_i}{\partial x_j},\ i, j = 1,2,3, \qquad (10)$$

And since **when** $b = 1$, $\mathbf{C}_1$**=К, then *Hooke* law (10)** corresponds to full *Taylor* series *d***u=К***d***x!**

The main paradox of *Navier-Cauchy-Lame* hypothsis is as follows. According to this hypothesis there are equivalent equalities (3) or (5), which result in equality to zero of curl components

$$\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = 0, i, j = 1,2,3 \text{ or } \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}, i, j = 1,2,3 \qquad (11)$$

Strains (4), built by *Navier-Cauchy-Lame* hypothesis, can be represented as follows:

$$\pi_{ji} = \lambda\delta_{ij} div\vec{u} + 2\mu\varepsilon_{ji}, \varepsilon_{ji} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) = \frac{\partial u_i}{\partial x_j} - \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}),$$

which, due to equalities (11) are transformed into formulae (10):

$$\pi_{ji} = \lambda\delta_{ij} div\vec{u} + \mu\varepsilon_{ji}, \varepsilon_{ji} = \frac{\partial u_i}{\partial x_j},\ i, j=1,2,3 \tag{12}$$

and are symmetric due to second equalities (11).

Again result is *Hooke* law in the form of (10)! **Main paradox is also in the fact that solutions of *Navier-Cauchy-Lame* equations**

$$\rho_0 \frac{\partial^2 \vec{u}}{\partial t^2} = \rho_0 \vec{F} + (\lambda + \mu) graddiv\vec{u} + \mu\Delta\vec{u} \tag{13}$$

must satisfy equalities (11). Let's study this issue. With this purpose we'll apply operation *rot* to equation (13):

$$\rho_0 \frac{\partial^2 rot\vec{u}}{\partial t^2} = \rho_0 rot\vec{F} + \mu\Delta rot\vec{u}\,, \tag{14}$$

since $rotgraddiv\vec{u} \equiv 0$.

Theorem 1. In nonstationary problems of elasticity theory $rot\vec{u} \neq 0$, **accordingly, *Navier-Cauchy-Lame* hypothesis (4) is not correct,** strain tensor **may not be symmetric.**

It is obvious, that for $rot\vec{F} \neq 0$ equation (14) has a nonzero solution $rot\vec{u} \neq 0$, therefore *Navier-Cauchy-Lame* hypothesis is not correct, which means that strain tensor needs to be defined by formula (10) with nonsymmetric strain tensor. Let $rot\vec{F} = 0$. Then equation (14) will take the following form

$$\rho_0 \frac{\partial^2 rot\vec{u}}{\partial t^2} = \mu\Delta rot\vec{u} \tag{15}$$

This wave equation has general nonzero solution

$$\begin{aligned} rot\vec{u} &= (\sin\pi x_1 \sin\pi x_2 \sin\pi x_3 \sin\sqrt{3\mu}\pi t)\vec{i}_1 + \\ &+ (\sin\pi x_1 \sin\pi x_2 \sin\pi x_3 \sin\sqrt{3\mu}\pi t)\vec{i}_2 + \\ &+ (\sin\pi x_1 \sin\pi x_2 \sin\pi x_3 \sin\sqrt{3\mu}\pi t)\vec{i}_3 \neq 0 \end{aligned} \tag{16}$$

accordingly, *Navier-Cauchy-Lame* hypothesis is incorrect again!

Let’s consider equation of elastic equilibrium, obtained by *Navier-Cauchy-Lame* hypothesis:

$$\rho_0\vec{F} + (\lambda+\mu) graddiv\vec{u} + \mu\Delta\vec{u} = 0 \qquad (17)$$

Curl operation (17) gives an elliptic equation relative to $rot\vec{u}$:

$$\rho_0 rot\vec{F} + \mu\Delta rot\vec{u} = 0 \qquad (18)$$

Theorem 2. **If** $rot\vec{F} \neq 0$, **equation (18) has a nonzero solution** $rot\vec{u} \neq 0$, **accordingly, in problems of elastic equilibrium, hypothesis of *Navier-Cauchy-Lame* is not correct, strain tensor may not be symmetric.**

The proof is straightforward.

Now let bulk forces are such that $rot\vec{F} = 0$. Then equation (18) will transform into homogenous elliptical equation

$$\Delta rot\vec{u} = 0,$$

which has a zero solution

$$rot\vec{u} = 0 \qquad (19)$$

Theorem 3. **In problems of elastic equilibrium, hypothesis of *Navier-Cauchy-Lame* is correct, if** $rot\vec{F} = 0$, **i.e.** $rot\vec{u}$ =0, **accordingly, strain tensor may be symmetrical. Displacement vector has a potential.**

Proof is already given in the form (19). Potential is introduced by formula

$$\vec{u} = grad\Phi, \qquad (20)$$

because $rot\vec{u} = rotgrad\Phi = 0$. And in this case strain tensor is defined as (10), symmetry takes place because of (19), which results in equalities (11).

So, according to theorems 1,2,3 strain tensor must have the following form (10): $\pi_{ji} = \lambda\delta_{ij} div\vec{u} + \mu\varepsilon_{ji}, \varepsilon_{ji} = \dfrac{\partial u_i}{\partial x_j}$, i, j=1,2,3.

According to *Navier-Cauchy-Lame* hypothesis, **forces acting upon a body** cause only **linear deformation E*d*x,** in the result a *distorted* expression is obtained, as said above,

$$du_i = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)dx_j, \quad d\mathbf{u}=\mathbf{E}d\mathbf{x},$$

having nothing in common with *Taylor* series (1) $d\mathbf{u}=\mathbf{K}d\mathbf{x}$. Rejection of *Navier-Cauchy-Lame* hypothesis means that if to base on formula of *Taylor* series

$$d\mathbf{u}=\mathbf{E}d\mathbf{x}+\frac{1}{2}[\mathrm{rot}\mathbf{u},d\mathbf{x}],$$

it is *necessary* to consider also forces of strains causing *Euler linear rotation*. With this purpose let's denote strains proportional to *Euler linear deformations* by π^{*}_{ij} -, strains proportional to *Euler linear* rotations by π^{**}_{ij}*:*

$$\pi^{*}_{ji}=\frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\quad \pi^{**}_{ji}=\frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right) \qquad (21)$$

By *Hooke* law total force will be

$$\pi_{ji}=\lambda\delta_{ij}div\vec{w}+\pi^{*}_{ji}+\pi^{**}_{ji}$$

Following substituting π^{*}_{ij} and π^{**}_{ij} obtained is

$$\pi_{ji}=\lambda\delta_{ij}div\vec{u}+\mu\frac{\partial u_i}{\partial x_j},\ i,j=1,2,3, \qquad (22)$$

where $\delta_{ij}=\begin{cases}0, i\neq j,\\ 1, i=j,\end{cases}$, $\lambda,\mu-$*Lame* coefficients.

Strain tensor in (22) is ***nonsymmetric*** $\pi_{ij}\neq\pi_{ji}, i\neq j$, because here equality to zero (11) component of vector rot**u** is not suggested initially. So, strain tensor (22) again matches tensors (12) and (10).

Nonsymmetry of strain tensor in continuum dynamics is proved in **Chapter 1**. Rigorous proof given there is easily transferred to deformed bodies in theory of elasticity as well. In monograph of *Lurie A.I.* /3/ et al. symmetry of strain tensor in theory of elasticity is established for the state of elastic equilibrium

$$\frac{\partial\vec{\pi}_x}{\partial x}+\frac{\partial\vec{\pi}_y}{\partial y}+\frac{\partial\vec{\pi}_z}{\partial z}+\rho\vec{F}=0 \qquad (23)$$

Similarly to *Sedov* and *Loytsynasky*, considered is moment of forces, acting upon body in volume τ with surface σ,

$$\iiint\limits_{\tau} [\vec{r}, \vec{F}\delta m] + \oint\!\!\oint\limits_{\sigma} [\vec{r}, \vec{\pi}_n \delta\sigma] = 0\,,$$

from which, due to arbitrariness of τ derived is relationship

$$\frac{\partial[\vec{r}, \vec{\pi}_{x]}}{\partial x} + \frac{\partial[\vec{r}, \vec{\pi}_y]}{\partial y} + \frac{\partial[\vec{r}, \vec{\pi}_z]}{\partial z} + [\vec{r}, \rho\vec{F}] = 0\,,$$

wherefrom, based on equation (23) *Lurie* et al. make a conclusion of symmetry of strain tensor in elastic medium. Criticism of such deductive method was given in **§8 Chapter 1**. Therefore, it shall be assumed, based on the above-mentioned facts, that tangential strains $\pi_{ij} \neq \pi_{ji}$, $j \neq i$ (in Cartesian coordinates $\pi_{xy} \neq \pi_{yx}$, $\pi_{zy} \neq \pi_{zy}$, $\pi_{xz} \neq \pi_{zx}$,) in the general case are not equal to each other**.** Equality of tangential strains – symmetry of strain tensor - at any point of continuum, i.e. execution of equalities (11) $\dfrac{\partial u_i}{\partial x_j} = \dfrac{\partial u_j}{\partial x_i}, i, j = 1,2,3,$ just means the fact that according to results of **§8 Chapter 1**, at the given point, moment of major (resultant) force matched with the major moment of forces, i.e. with resultant of moments of forces.

Thus, it is natural that there is a long-lasting need of revising equations of dynamics of deformed solids, in other words, equations of *Navier-Cauchy-Lame.*

§4. Paradoxes of *Hooke* law for symmetric strain tensor *Navier-Cauchy-Lame*

Let's, for example, consider equations of elastic equilibrium (17) **§3** for a two-dimensional case, when force $\vec{F} = 0\vec{i}_1 + 0\vec{i}_2 + F_3\vec{i}_3$ is perpendicular to plane (x_1, x_2), and will represent (17) in projections

$$(\lambda + \mu)\frac{\partial}{\partial x_k} div\vec{u} + \mu\Delta u_k = 0, k = 1,2,3\,, \tag{24}$$

or in more details for a two-dimensional problem

$$\lambda \frac{\partial}{\partial x_k}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) + \mu \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right) = 0, k = 1,2 \tag{25}$$

For infinite number of displacements, i.e. solutions of system of equations (25), **symmetric tangential strains of *Navier-Cauchy-Lame* hypothesis have zero values**, i.e. in all points of the body are zero

$$\pi_{ij} = \pi_{ji} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 0, i, j = 1,2,$$

while normal strains are different from zero $\pi_{ii} = 2\mu\varepsilon_{ii} \neq 0$.

Let's limit to giving a small list of displacements, in which this fact is actual (for shortness, denoted as $u_1 \equiv u, u_2 \equiv v, x_1 \equiv x, x_2 \equiv y, u_3 \equiv w, x_3 \equiv z$):

1) $u=F(\sin k_1 x \cos k_1 y)$, $v=F(-\cos k_1 x \sin k_1 y)$,

2) $u= U(\sin k_2 x \cos k_2 y - \cos k_2 x \sin k_2 y)$,

$v= U(\sin k_2 x \cos k_2 y - \cos k_2 x \sin k_2 y)$,

3) $u=W(-\cos k_3 x \sin k_3 y)$, $v=W(\sin k_3 x \cos k_3 y)$,

4) $u=Q(\sin k_4 x \ \sin k_4 y)$, $v=Q(\cos k_4 x \ \cos k_4 y)$,

5) $u=T(\sin k_5 x \sin k_5 y)$, $v=T(\cos k_5 x \ \cos k_5 y)$, (26)

6) $u=M(\sin k_6 x \sin k_6 y + \cos k_6 y \cos k_6 x)$,

$v=M(\sin k_6 x \sin k_6 y + \cos k_6 y \cos k_6 x)$,

7) $u = S(e^{k_7(x+y)}), \quad v = -S(e^{k_7(x+y)})$,

for three-dimensional displacements when there are no bulk forces

8) u=D($(e^{k_8 y} - e^{k_8 z}) e^{k_8 x}$), v=D($(e^{k_8 z} - e^{k_8 x})\ e^{k_8 y}$), (27)

w=D($(e^{k_8 x} - \ e^{k_8 y}) e^{k_8 z}$), where coefficients $k_i = const$, $i = 1,2,3,4,5,6,7,8,$ are selected arbitrary from infinite interval $-\infty < k_i < +\infty$. Standing here differentiable functions F,U,W,Q,T,M, S,D are also chosen arbitrary. Obviously, from the given list it is possible to form any new linear combinations of *u=F+U, v=F+U* etc. type. Fields 1-7 in (26) correspond to plane displacements, and are solutions of equations (25), since they transform divergence into

zero $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$, fields of displacement 8) in (27) turn three-dimensional divergence into zero $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$, symmetric tangential strains are equal to zero $\pi_{ji}=\mu(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})=0, i\neq j, \forall(i,j)$, normal strains are nonzero $\pi_{ii}=2\mu\varepsilon_{ii}\neq 0, i=1,2,3$. It is necessary to note that partial solutions (26), (27) of elastic equilibrium equations (24), are also partial solutions for equations with **nonsymmetric** strains **tensor** $\lambda\frac{\partial}{\partial x_k}div\vec{u}+\mu\Delta u_k=0, k=1,2,3$, but here, *nonsymmetric* tangential strains are nonzero $\pi_{ji}=\mu\frac{\partial u_i}{\partial x_j}\neq 0, \pi_{ij}\neq\pi_{ji}, i\neq j$, which is physical enough.

§5. Equations of elasticity theory for *nonsymmetric* strain tensor

Elasticity theory equations in the form of *Navier-Cauchy-Lame* for symmetric strain tensor are given in /1/-/3/:

$$\rho_0\frac{\partial^2\vec{u}}{\partial t^2}=\rho_0\vec{F}+(\lambda+\mu)graddiv\vec{u}+\mu\Delta\vec{u}, \qquad (1)$$

$\vec{u}=u\vec{i}+v\vec{j}+w\vec{k}$ - displacement vector, $u_1\equiv u, u_2\equiv v, u_3\equiv w$.

Above it was justified, that if forces acting upon particles of deformed body move them to

$$d\mathbf{u}=\mathbf{K}d\mathbf{x},$$

then appropriate components of strain tensor must, in accordance with *Hooke* law, be proportional to components of **nonsymmetric displacement tensor К**.

Hooke law for nonsymmetric strain tensor was given in **§2**

$$\pi_{ji}=\lambda\delta_{ij}div\vec{u}+\mu\varepsilon_{ji}, \quad \varepsilon_{ji}=\frac{\partial u_i}{\partial x_j}, i,j=1,2,3 \qquad (2)$$

Dynamics equations are result from substituting (2) into equations

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \rho_0 F_i + \sum_{j=1}^{3} \frac{\partial \pi_{ji}}{\partial x_j}, i = 1,2,3$$

and for nonsymmetric strain tensor (2) it is derived

$$\rho_0 \frac{\partial^2 \vec{u}}{\partial t^2} = \rho_0 \vec{F} + \lambda graddiv\vec{u} + \mu\Delta\vec{u} \qquad (3)$$

Projections (3) on coordinate axis form a system of three scalar hyperbolic-type equations

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \rho_0 F_x + \lambda \frac{\partial}{\partial x}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}),$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \rho_0 F_y + \lambda \frac{\partial}{\partial y}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}), \quad (4)$$

$$\rho_0 \frac{\partial^2 w}{\partial t^2} = \rho_0 F_z + \lambda \frac{\partial}{\partial z}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + \mu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2})$$

Further on given are numerical methods of solving problems of elasticity theory in displacements.

Note. Consistency conditions of components of nonsymmetric displacement tensor are

$$\frac{\partial^2 \varepsilon_{\nu i}}{\partial x_j \partial x_\mu} + \frac{\partial^2 \varepsilon_{i\nu}}{\partial x_\mu \partial x_j} + \frac{\partial^2 \varepsilon_{\mu j}}{\partial x_i \partial x_\nu} + \frac{\partial^2 \varepsilon_{j\mu}}{\partial x_\nu \partial x_i} -$$

$$- \frac{\partial^2 \varepsilon_{\mu i}}{\partial x_j \partial x_\nu} - \frac{\partial^2 \varepsilon_{i\mu}}{\partial x_\nu \partial x_j} - \frac{\partial^2 \varepsilon_{\nu j}}{\partial x_i \partial x_\nu} - \frac{\partial^2 \varepsilon_{j\nu}}{\partial x_\nu \partial x_i} = 0$$

§6. Explicit scheme

Let's state the idea of building such a scheme in the example of two-dimensional initially-boundary problem

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \rho_0 F_x + (\lambda + \mu^0) \frac{\partial}{\partial x}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}),$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \rho_0 F_y + (\lambda + \mu^0)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),$$

$$u\Big|_{t=t_0} = d_0(x,y), \quad \frac{\partial u}{\partial t}\Big|_{t=t_0} = d(x,y), u\Big|_S = \varphi(x,y,t),$$

$$v\Big|_{t=t_0} = \tilde{d}_0(x,y), \quad \frac{\partial v}{\partial t}\Big|_{t=t_0} = \tilde{d}(x,y), v\Big|_S = \tilde{\varphi}(x,y,t)$$

Here, parameter $\mu^0 = \mu$ for symmetric strain tensor and $\mu^0 = 0$ for nonsymmetric strain tensor. The scheme has the following form

$$\rho_0 \frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{\tau^2} = \rho_0 F_x + (\lambda + \mu^0)\{\frac{1}{\hbar_{xi}}(u_x^n - u_{\bar{x}}^n) +$$

$$+\frac{1}{h_{xi+1} + h_{xi}}\left(\frac{v_{i+1j+1}^n - v_{i+1j-1}^n}{h_{yj+1} + h_{yj}} - \frac{v_{i-1j+1}^n - v_{i-1j-1}^n}{h_{yj+1} + h_{yj}}\right)\} + \mu[\frac{1}{\hbar_{xi}}(u_x^n - u_{\bar{x}}^n) + \frac{1}{\hbar_{yj}}(u_y^n - u_{\bar{y}}^n)],$$

$$\rho_0 \frac{v_{ij}^{n+1} - 2v_{ij}^n + v_{ij}^{n-1}}{\tau^2} = \rho_0 F_y + (\lambda + \mu^0)\{\frac{1}{\hbar_{yj}}(v_y^n - v_{\bar{y}}^n) +$$

$$+\frac{1}{h_{yj+1} + h_{yj}}\left(\frac{u_{i+1j+1}^n - u_{i-1j+1}^n}{h_{xi+1} + h_{xi}} - \frac{u_{i+1j-1}^n - u_{i-1j-1}^n}{h_{xi+1} + h_{xi}}\right)\} + \mu[\frac{1}{\hbar_{xi}}(v_x^n - v_{\bar{x}}^n) + \frac{1}{\hbar_{yj}}(v_y^n - v_{\bar{y}}^n)],$$

$$i = 1,\ldots,N_x - 1, \; j = 1,\ldots,N_y - 1$$

Initial conditions are specified

$$\Omega_h = \{x_i, i = 1,\ldots,N_x - 1; y_j, \; j = 1,\ldots,N_y - 1\}:$$

$$u_{ij}^0 = d_{0ij}, v_{ij}^0 = \tilde{d}_{0ij}, u_{ij}^1 = d_{0ij} + \tau d_{ij}, v_{ij}^1 = \tilde{d}_{0ij} + \tau\tilde{d}_{ij},$$

and boundary conditions of the first kind

$$S_h = \{i = 0, i = N_x, j = \overline{0, N_y}; j = 0, j = N_y, i = \overline{0.N_x}\}, \bar{\Omega}_h = \Omega_h \bigcup S_h,$$

$$x = 0, u_{0j}^{n+1} = \varphi_{0j}^{n+1}, v_{0j}^{n+1} = \tilde{\varphi}_{0j}^{n+1}, \; j = \overline{0, N_y},$$

$$x = a_1, u_{N_x j}^{n+1} = \varphi_{N_x j}^{n+1}, v_{N_x j}^{n+1} = \tilde{\varphi}_{N_x j}^{n+1}, \; j = \overline{0, N_y},$$

$$y = 0, u_{i0}^{n+1} = \varphi_{i0}^{n+1}, v_{i0}^{n+1} = \phi_{i0}^{n+1}, i = \overline{0, N_x},$$

$$y = a_2, u_{iN_y}^{n+1} = \varphi_{iN_y}^{n+1}, v_{iN_y}^{n+1} = \phi_{iN_y}^{n+1}, i = \overline{0, N_x}$$

errors of approximation are

$$\psi_{uij}^{n} = O(\tau^2) + O(\hbar_{xi}^2) + O(h_{xi+1} - h_{xi}) + O(\hbar_{yj}^2) + O(h_{yj+1} - h_{yj}),$$

$$\psi_{vij}^{n} = O(\tau^2) + O(\hbar_{xi}^2) + O(h_{xi+1} - h_{xi}) + O(\hbar_{yj}^2) + O(h_{yj+1} - h_{yj}),$$

$$i = 1,\ldots,N_x - 1,\ j = 1,\ldots,N_y - 1;$$

Condition of stability is the *Courant* criterion

$$\tau^2 \mu[\frac{2}{h_{xi+1} + h_{xi}}(\frac{1}{h_{xi+1}} + \frac{1}{h_{xi}}) + \frac{2}{h_{yj+1} + h_{yj}}(\frac{1}{h_{yj+1}} + \frac{1}{h_{yj}})] \le 1,$$

$$i = 1,\ldots,N_x - 1,\ j = 1,\ldots,N_y - 1.$$

Note. Combined coefficient $\lambda + \mu^0$ is included with a special purpose of checking symmetry conditions of strain tensor $\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}, i, j$=1,2,3 for $\mu^0 = \mu$ in numerical computation of specific problems.

§7. Solution of three-dimensional problems of elasticity theory on a uniform mesh

For three-dimensional equations

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \rho_0 F_x + \lambda \frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}),$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \rho_0 F_y + \lambda \frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}),$$

$$\rho_0 \frac{\partial^2 w}{\partial t^2} = \rho_0 F_x + \lambda \frac{\partial p}{\partial z} + \mu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}),$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=p$$

specified are meshes, in their nodes initial conditions

$$\Omega_h=\{x_i,i=1,\ldots,N_x-1;y_j,j=1,\ldots,N_y-1;z_m,m=1,\ldots,N_z-1\}:$$

$$u_{ijm}^0=d_{0ijm},v_{ijm}^0=\tilde{d}_{0ijm},w_{ijm}^0=\tilde{\tilde{d}}_{0ijm},u_{ijm}^1=d_{0ijm}+\tau d_{ijm},$$

$$v_{ijm}^1=\tilde{d}_{0ijm}+\tau\tilde{d}_{ijm},w_{ijm}^1=\tilde{\tilde{d}}_{0ijm}+\tau\tilde{\tilde{d}}_{ijm},$$

$$i=\overline{1,N_x-1},j=\overline{1,N_y-1},m=\overline{1,N_z-1}$$

and boundary conditions of the first kind

$$S_h=\{i=0,i=N_x,j=\overline{0,N_y},m=\overline{0,N_z};j=0,j=N_y,$$

$$i=\overline{0,N_x},m=\overline{0,N_z};m=0,m=N_z,j=\overline{0,N_y},i=\overline{0,N_x}\},$$

$$\overline{\Omega}_h=\Omega_h\bigcup S_h;$$

$$x=0,u_{0jm}^{n+1}=\varphi_{0jm}^{n+1},v_{0jm}^{n+1}=\tilde{\varphi}_{0jm}^{n+1},w_{0jm}^{n+1}=\tilde{\tilde{\varphi}}_{0jm}^{n+1},$$

$$x=a_1,u_{N_xjm}^{n+1}=\varphi_{N_xjm}^{n+1},v_{N_xjm}^{n+1}=\tilde{\varphi}_{N_xjm}^{n+1},w_{N_xjm}^{n+1}=\tilde{\tilde{\varphi}}_{N_xjm}^{n+1},$$

$$j=\overline{0,N_y},m=\overline{0,N_z},\ y=0,u_{i0m}^{n+1}=\varphi_{i0m}^{n+1},v_{i0m}^{n+1}=\tilde{\varphi}_{i0m}^{n+1},w_{i0m}^{n+1}=\tilde{\tilde{\varphi}}_{i0m}^{n+1},$$

$$y=a_2,u_{iN_ym}^{n+1}=\varphi_{iN_ym}^{n+1},v_{iN_ym}^{n+1}=\tilde{\varphi}_{iN_ym}^{n+1},w_{iN_ym}^{n+1}=\tilde{\tilde{\varphi}}_{iN_ym}^{n+1},$$

$$i=\overline{0,N_x},m=\overline{0,N_z},$$

$$z=0,u_{ij0}^{n+1}=\varphi_{ij0}^{n+1},v_{ij0}^{n+1}=\tilde{\varphi}_{ij0}^{n+1},w_{ij0}^{n+1}=\tilde{\tilde{\varphi}}_{ij0}^{n+1},$$

$$z=a_3,u_{ijN_z}^{n+1}=\varphi_{ijN_z}^{n+1},v_{ijN_z}^{n+1}=\tilde{\varphi}_{ijN_z}^{n+1},w_{ijN_z}^{n+1}=\tilde{\tilde{\varphi}}_{ijN_z}^{n+1},$$

$$i=\overline{0,N_x},\ j=\overline{0,N_y},$$

global iterative process is built as follows:

$$u_{ijm}^{n+1,k}=F_{uijm}^n+\frac{\tau^2}{\rho_0}\lambda\frac{p_{i+1jm}^{n+1,k}-p_{i-1jm}^{n+1,k}}{2h_x},$$

$$v_{ijm}^{n+1,k}=F_{vijm}^{n}+\frac{\tau^2}{\rho_0}\lambda\frac{p_{ij+1m}^{n+1,k}-p_{ij-1m}^{n+1,k}}{2h_y},$$

$$w_{ijm}^{n+1,k}=F_{wijm}^{n}+\frac{\tau^2}{\rho_0}\lambda\frac{p_{ijm+1}^{n+1,k}-p_{ijm-1}^{n+1,k}}{2h_z},$$

$$\mathbf{1\le \boldsymbol{i}\le \boldsymbol{N}_{\boldsymbol{x}}-1{,}1\le \boldsymbol{j}\le \boldsymbol{N}_{\boldsymbol{y}}-1{,}1\le \boldsymbol{m}\le \boldsymbol{N}_{\boldsymbol{z}}-1},$$

$$(1+A_{ijm}\theta)\frac{p_{ijm}^{n+1,k+1}-p_{ijm}^{n+1,k}}{\theta}=\frac{u_{i+1jm}^{n+1,k}-u_{i-1jm}^{n+1,k}}{2h_x}+\frac{v_{ij+1m}^{n+1,k}-v_{ij-1m}^{n+1,k}}{2h_y}+$$

$$+\frac{w_{ijm+1}^{n+1,k}-m_{ijm-1}^{n+1,k}}{2h_z}-p_{ijm}^{n+1,k},$$

$$\mathbf{1\le \boldsymbol{i}\le \boldsymbol{N}_{\boldsymbol{x}}-1{,}1\le \boldsymbol{j}\le \boldsymbol{N}_{\boldsymbol{y}}-1{,}1\le \boldsymbol{m}\le \boldsymbol{N}_{\boldsymbol{z}}-1},$$

$$(1+A_{N_xjm}\theta)\frac{p_{N_xjm}^{n+1,k+1}-p_{N_xjm}^{n+1,k}}{\theta}=\frac{\varphi_{N_xjm}^{n+1}-u_{N_x-1jm}^{n+1,k}}{h_x}+$$

$$+\left(\frac{\partial\varphi}{\partial y}\right)_{N_xjm}^{n+1}+\left(\frac{\partial\varphi}{\partial z}\right)_{N_xjm}^{n+1}-p_{N_xjm}^{n+1,k},$$

$$(1+A_{0jm}\theta)\frac{p_{0jm}^{n+1,k+1}-p_{0jm}^{n+1,k}}{\theta}=\frac{u_{1jm}^{n+1,k}-\varphi_{0jm}^{n+1}}{h_x}+$$

$$+\left(\frac{\partial\varphi}{\partial y}\right)_{0jm}^{n+1}+\left(\frac{\partial\varphi}{\partial z}\right)_{0jm}^{n+1}-p_{0jm}^{n+1,k},1\le j\le N_y-1{,}1\le m\le N_z-1,$$

$$(1+A_{iN_ym}\theta)\frac{p_{iN_ym}^{n+1,k+1}-p_{iN_ym}^{n+1,k}}{\theta}=\left(\frac{\partial\varphi}{\partial x}\right)_{iN_ym}^{n+1}+\frac{\varphi_{iN_ym}^{n+1}-v_{iN_y-1m}^{n+1,k}}{h_y}+$$

$$+\left(\frac{\partial\varphi}{\partial z}\right)_{iN_y m}^{n+1} - p_{iN_y m}^{n+1,k},$$

$$(1+A_{i0m}\theta)\frac{p_{i0m}^{n+1,k+1}-p_{i0m}^{n+1,k}}{\theta} = \left(\frac{\partial\varphi}{\partial x}\right)_{i0m}^{n+1} + \frac{v_{i1m}^{n+1}-\varphi_{i0m}^{n+1,k}}{h_y} +$$

$$+\left(\frac{\partial\varphi}{\partial z}\right)_{i0m}^{n+1} - p_{i0m}^{n+1,k}, 1\le i\le N_x-1, 1\le m\le N_z-1,$$

$$(1+A_{ijN_z}\theta)\frac{p_{ijN_z}^{n+1,k+1}-p_{ijN_z}^{n+1,k}}{\theta} = \left(\frac{\partial\varphi}{\partial x}\right)_{ijN_z}^{n+1} + \frac{\varphi_{ijN_z}^{n+1}-w_{ijN_z-1}^{n+1,k}}{h_z} +$$

$$+\left(\frac{\partial\varphi}{\partial y}\right)_{ijN_z}^{n+1} - p_{ijN_z}^{n+1,k},$$

$$(1+A_{ij0}\theta)\frac{p_{ij0}^{n+1,k+1}-p_{ij0}^{n+1,k}}{\theta} = \left(\frac{\partial\varphi}{\partial x}\right)_{ij0}^{n+1} + \frac{w_{ij1}^{n+1}-\varphi_{ij0}^{n+1,k}}{h_z} +$$

$$+\left(\frac{\partial\varphi}{\partial y}\right)_{ij0}^{n+1} - p_{ij0}^{n+1,k}, 1\le i\le N_x-1, 1\le j\le N_y-1,$$

where

$$A_{ijm} = 1+\frac{\tau^2}{\rho_0}\lambda\left\{\frac{1}{2h_x}\left[\frac{1}{2h_x}\frac{1-sign(i-N_x+1.5)}{2}+\right.\right.$$

$$\left.+\frac{1}{2h_x}\frac{1+sign(i-1.5)}{2}\right]+\frac{1}{2h_y}\left[\frac{1}{2h_y}\frac{1-sign(j-N_y+1.5)}{2}+\right.$$

$$\left.+\frac{1}{2h_y}\frac{1+sign(j-1.5)}{2}\right]+\frac{1}{2h_z}\left[\frac{1}{2h_z}\frac{1-sign(m-N_z+1.5)}{2}+\right.$$

$$\left.\left.+\frac{1}{2h_z}\frac{1+sign(m-1.5)}{2}\right]\right\},$$

$$i=1,\ldots,N_x-1,\ j=1,\ldots,N_y-1,\ m=1,\ldots,N_z-1;$$

$$A_{N_x jm}=1+\frac{\tau^2}{\rho_0}\lambda\frac{1}{2h_x^2},A_{0jm}=1+\frac{\tau^2}{\rho_0}\lambda\frac{1}{2h_x^2},$$

$$j=1,\ldots,N_y-1,m=1,\ldots,N_z-1;$$

$$A_{iN_y m}=1+\frac{\tau^2}{\rho_0}\lambda\frac{1}{2h_y^2},A_{i0m}=1+\frac{\tau^2}{\rho_0}\lambda\frac{1}{2h_y^2},$$

$$i=1,\ldots,N_x-1,m=1,\ldots,N_z-1;$$

$$A_{ijN_z}=1+\frac{\tau^2}{\rho_0}\lambda\frac{1}{2h_z^2},A_{ij0}=1+\frac{\tau^2}{\rho_0}\lambda\frac{1}{2h_z^2},$$

$$i=1,\ldots,N_x-1, j=1,\ldots,N_y-1$$

Parameter $\theta>0$ is selected from interval $0<\theta\le 1/2$.

Process of calculations is stopped on such number of iterations k^*, with which the following inequality is true

$$\max_{0\le i\le N_x,0\le i\le N_y,0\le m\le N_z}|(1+A_{ijm}\theta)\frac{p_{ijm}^{n+1,k^*+1}-p_{ijm}^{n+1,k^*}}{\theta}|\le\varepsilon,\quad \varepsilon\approx 0,\varepsilon\ne 0$$

While observing this criterion the last approximations

$p_{ijm}^{n+1,k^*},u_{ijm}^{n+1,k^*},v_{ijm}^{n+1,k^*},w_{ijm}^{n+1,k^*}$, are taken as solutions

$$p_{ijm}^{n+1,k^*}\approx p_{ijm}^{n+1},u_{ijm}^{n+1,k^*}\approx u_{ijm}^{n+1},v_{ijm}^{n+1,k^*}\approx v_{ijm}^{n+1},w_{ijm}^{n+1,k^*}\approx w_{ijm}^{n+1},$$

$i=0,\ldots,N_x, j=0,\ldots,N_y, m=0,\ldots,N_z$.

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